The Choquet Simplex

In this lecture we study the structure of the set of T-invariant probability measures $\mathcal{M}(X, T)$ on a compact metric space X.

- $\mathcal{M}(X, T)$ is invariant under taking convex combinations, and therefore a simplex.
- It is called the Choquet simplex. The actual definition of a Choquet simplex is that it is a compact, metrizable, convex set in which every element can be decomposed uniquely as convex combination of extremal points.
- A point x in a simplex is extremal if the only way of writing it as a convex combination

$$x = \sum_i \alpha_i x_i \qquad \alpha_i \in [0, 1], \quad \sum_i \alpha_i = 1$$

  is th trivial way: $x = 1 \cdot x$.
- Let $\mathcal{M}_{erg}(X, T)$ be the subset of $\mathcal{M}(X, T)$ of ergodic T-invariant measures.

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Lemma: The ergodic measures are the extremal points of the Choquet simplex.

Proof: First assume that μ is not ergodic. Hence there is a T-invariant set A such that $0 < \mu(A) < 1$. Define

$$\mu_1(B) = \frac{\mu(B \cap A)}{\mu(A)} \quad \text{and} \quad \mu_2(B) = \frac{\mu(B \setminus A)}{\mu(X \setminus A)}.$$

Then $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for $\alpha = \mu(A) \in (0,1)$ so μ is not an extremal point.

Conversely, suppose that μ is ergodic, and $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for some $\alpha \in (0,1)$ and $\mu_1, \mu_2 \in \mathcal{M}(X, T)$. Then $\mu_1 \ll \mu$ and also $\mu_2 \ll \mu$. But this implies that $\mu_1 = \mu_2 = \mu$, so the convex combination is trivial and μ must be extremal.

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Lemma: The Choquet simplex $\mathcal{M}(X, T)$ is a compact subset of $\mathcal{M}(X)$ w.r.t. weak* topology.

Proof: Suppose $\{\mu_n\} \subset \mathcal{M}(X, T)$, then by the compactness of $\mathcal{M}(X)$ in the weak* topology there is $\mu \in \mathcal{M}(X)$ and a subsequence $(n_i)_i$ such that $\int f \, d\mu_{n_i} \to \int f \, d\mu$ as $i \to \infty$ for every continuous function $f : X \to \mathbb{R}$.

It remains to show that μ is T-invariant, but this follows from continuity of $f \circ T$ and

$$\int f \circ T \, d\mu = \lim_i \int f \circ T \, d\mu_{n_i} = \lim_i \int f \, d\mu_{n_i} = \int f \, d\mu.$$

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The following fundamental theorem implies that for checking the properties of any measure $\mu \in \mathcal{M}(X, T)$, it suffices to verify the properties for ergodic measures:

Theorem: (Ergodic Decomposition): For every $\mu \in \mathcal{M}(X, T)$, there is a measure ν on the space of ergodic measures such that $\nu(\mathcal{M}_{erg}(X, T)) = 1$ and

$$\mu(B) = \int_{\mathcal{M}_{erg}(X,T)} m(B)\, d\nu(m)$$

for every Borel set B.

E.g. If $\mu = \sum_{i=1}^{N} \alpha_i m_i$, $m_i \in \mathcal{M}_{erg}$, then $\nu = \sum_{i=1}^{N} \alpha_i \delta_{m_i}$.

The Poulsen Simplex

Definition: The simplex $\mathcal{M}(X, T)$ of T-invariant probability measures is called a Poulsen simplex if it is not degenerate, but the extremal points (i.e. $\mathcal{M}_{erg}(X, T)$) lie dense in $\mathcal{M}(X, T)$.

This definition shows what a enormous and complicated space the Choquet simplex can be. And it is a reality for many dynamical systems, as we will demonstrate, as an example, for the doubling map.

Proposition: The Choquet simplex of the doubling map $T : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto 2x \bmod 1$, is a Poulsen simplex.

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Proof: First note that an equidistribution on periodic orbits is an ergodic measure. Therefore it suffices to show that the equidistributions lie dense in the Choquet simplex $\mathcal{M}(\mathbb{S}^1, T)$.

Claim: for every $\delta > 0$, there is $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ and $x \in \mathbb{S}^1$ there is $y \in \mathbb{S}^1$ such that

- $|T^k(x) - T^k(y)| < \delta$ for all $0 \leq k < n$;
- $T^{n+N}(y) = y$.

Proof of claim: Take N so large that $2^{-N} < \delta$ and $n \in \mathbb{N}$, $x \in \mathbb{S}^1$ arbitrary. Take y in the same dyadic interval J of generation $n + N$ as x, so that $T^{n+N}(y) = y$. Since $T^{n+N}(J) = \mathbb{S}^1$ this is possible. Also

$$|T^k(x) - T^k(y)| \leq |T^k(J)| \leq |T^n(J)| = 2^{-N} < \delta$$

for all $0 \leq k < n$. The claim is proved.

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- Let $\mu \in \mathcal{M}$ and $m \in \mathbb{N}$ be arbitrary.
- Let x be a typical point for μ, i.e., for every $f \in C(X)$ with $\|f\|_\infty \leq m$, we have $\lim_n \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int_X f \, d\mu$.
- We can find a finite collection $f_j \subset C(X)$ s.t. $\forall \, f \in C(X)$ with $\|f\|_\infty \leq m$, we have $| \int_X f_j \, d\mu - \int_X f \, d\mu| < \frac{1}{m}$ for some f_j.
- Since f_j is continuous, it is uniformly continuous on the compact space X. Hence we can find $\delta > 0$ such that $|x - y| < \delta$ implies $|f_j(x) - f_j(y)| < 1/m$ for all j. For this $\delta > 0$, take $N \in \mathbb{N}$ as in the claim.
- Take $n > 2Nm^2$ so large that for each f_j, $|\frac{1}{n} \sum_{k=0}^{n-1} f_j \circ T^k(x) - \int_X f_j \, d\mu| < \frac{1}{m}$.

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Now find the $n+N$-periodic point y close to x as in the claim. Let ν_m be the equidistribution on the orbit of y. Then

$$\int_X f\, d\nu_m = \frac{1}{n+N}\sum_{k=0}^{n+N-1} f\circ T^k(y)$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} f_j\circ T^k(y)$$

$$- \frac{N}{n(n+N)}\sum_{k=0}^{n-1} f_j\circ T^k(y) + \frac{1}{n+N}\sum_{k=n}^{n+N-1} f_j\circ T^k(y)$$

These last two terms together are

$$\leq \frac{2N}{n+N}\|f\|_\infty \leq \frac{2Nm}{n} < \frac{1}{m}$$

in absolute value because $n > 2Nm^2$.

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Therefore

$$\begin{aligned}
& |\int_X f\ d\mu - \int_X f\ d\nu_m| \\
\text{def. of } f_j \& \nu_m & \leq |\int_X f_j\ d\mu - \frac{1}{n+N}\sum_{k=0}^{n+N-1} f \circ T^k(y)| + \frac{1}{m} \\
x \text{ is typical} & \leq |\frac{1}{n}\sum_{k=0}^{n-1} f_j \circ T^k(x) - \frac{1}{n+N}\sum_{k=0}^{n+N-1} f_j \circ T^k(y)| + \frac{2}{m} \\
\text{previous slide} & \leq \frac{1}{n}|\sum_{k=0}^{n-1} f_j \circ T^k(x) - f_j \circ T^k(y)| + \frac{3}{m} \\
\text{unif. continuity} & \leq \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{m} + \frac{3}{m} = \frac{4}{m}.
\end{aligned}$$

Hence the sequence (ν_m) converges to μ in the weak* topology.