Poincaré Recurrence

Given a dynamical system (X, T), it is sometimes useful to consider the first return map to a subset $Y \subset X$.

Define the first return time to a set Y as

$$\tau_Y = \min\{n \geq 1 : T^n(x) \in Y\}.$$

The first return map or induced map to Y is

$$T_Y : Y \to Y, \quad T_Y(y) = T^{\tau(y)}(y).$$

The Poincaré Recurrence Theorem: If (X, T, μ) is a measure preserving system with $\mu(X) = 1$, then for every measurable set $Y \subset X$ of positive measure, μ-a.e. $y \in Y$ returns to Y, i.e., $\tau(y) < \infty$ μ-a.e.

Poincaré Recurrence

Proof:

Assume by contradiction that

$$Y' = \{x \in Y : T^n(x) \notin Y \text{ for all } n \geq 1\}$$

has measure $\mu(Y') > 0$.

As μ is invariant, $\mu(T^{-i}(Y')) = \mu(Y') > 0$ for all $i \geq 0$. On the other hand,

$$1 = \mu(X) \geq \mu(\cup_i T^{-i}(Y')),$$

so there must be overlap in the backward iterates of Y'.

That is: there are $0 \leq i < j$ such that $\mu(T^{-i}(Y') \cap T^{-j}(Y')) > 0$. Take the i-th iterate and find

$$\mu(Y' \cap T^{i-j}(Y')) \geq \mu(T^{-i}(Y') \cap T^{-j}(Y')) > 0,$$

so there is $z \in Y'$ such that $T^{j-i}(z) \in Y' \subset Y$, contradicting the definition of Y'. This concludes the proof.

Poincaré Recurrence

Definition: A system $(X, T, \mathcal{B}, \mu)$ is called conservative if for every set $A \in \mathcal{B}$ with $\mu(A) > 0$, there is $n \geq 1$ such that $\mu(T^n(A) \cap A) > 0$. The Poincaré Recurrence Theorem thus implies that probability measure preserving systems are conservative.

If not conservative, then the system is called dissipative. It is called totally dissipative if for every set $A \in \mathcal{B}$,

$$\mu(\{x \in A : T^n(x) \in A \text{ infinitely often}\}) = 0.$$

Kac's Lemma

The next result quantifies the expected value of the first return time τ to $Y \subset X$.

Kac' Lemma: Let (X, T) preserve an ergodic measure μ. Take $Y \subset X$ measurable such that $\mu(Y) > 0$, and let $\tau : Y \to \mathbb{N}$ be the first return time to Y. Take $Y \subset X$ measurable such that $\mu(Y) > 0$. Then

$$\int_Y \tau d\mu = \sum_{n \geq 1} n\mu(Y_n) = \mu(X)$$

for $Y_n := \{y \in Y : \tau(y) = n\}$.

Exercise 7.1: Extend τ to $\tau : X \to \mathbb{N}$ as the first hitting time on X:

$$\tau(x) = \min\{n \geq 1 : T^n(x) \in Y\}.$$

Show that $\int_X \tau(x)\, d\mu = \mu(Y)^{-1}$.

Kac's Lemma

Proof of Kac' Lemma Set

$$U_n = \{x \in Y : T_Y^n(x) \text{ is undefined }\}.$$

By Poincaré Recurrence, $\mu(U_1) = 0$. Also

$$U_{n+1} = \cup_{k \geq 1} T^{-k}(U_n) \cap Y_k$$

for $Y_k = \{y \in Y : \tau(y) = k\}$. It follows by induction that $\mu(U_n) = 0$ for all $n \geq 1$. Hence $\mu(\cup_n U_n) = 0$ as well: μ-a.e. $y \in Y$ returns to Y infinitely often.

Let $A = \{x \in X : T^n(x) \in Y \text{ infinitely often}\}$. Then $A = T^{-1}(A)$. and $\mu(A) > \mu(Y) > 0$. By ergodicity, $\mu(A^c) = 0$.

Kac's Lemma

Next define $L_0 = Y$, $L_1 = T^{-1}(L_0) \setminus Y$ and recursively $L_{n+1} = T^{-1}(L_n) \setminus Y$. In other words:

$$L_n = \{x \in X : T^n(x) \in Y \text{ and } T^k(x) \notin Y \text{ for } 0 \leq k < n\}.$$

Clearly all the L_ns are pairwise disjoint, and

$$\mu(X) \geq \sum_{n \geq 0} \mu(L_n) \geq \mu(A) = \mu(X)$$

by the previous slide.

Furthermore, $T^{-1}(L_n)$ is the disjoint union of L_{n+1} and Y_{n+1}. By T-invariance of μ it follows that

$$\mu(L_n) = \mu(L_{n+1}) + \mu(Y_{n+1}).$$

Kac's Lemma

Therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} n\mu(Y_n) &= \sum_{n=0}^{\infty} (n+1)\mu(Y_{n+1}) \\
&= \sum_{n=0}^{\infty} (n+1)(\mu(L_n) - \mu(L_{n+1})) \\
&= \sum_{n=0}^{\infty} \mu(L_n) + \underbrace{n\mu(L_n) - (n+1)\mu(L_{n+1})}_{\text{telescopes to } 0} \\
&= \sum_{n=0}^{\infty} \mu(L_n) = \mu(X).
\end{aligned}$$

This proves Kac' Lemma.

Induced Tranformations

Definition: We say that a transformation T is non-singular w.r.t. a measure μ if $\mu(A) = 0$ implies that $\mu(T^{-1}(A)) = 0$.

Hence measures-preserving maps are always non-singular, and so are e.g. differential maps w.r.t. Lebesgue measure.

But T is not non-singular w.r.t. the Dirac measure δ_x, unless x is a fixed point.

Induced Tranformations

Kac's Lemma effectively combines a measure preserving system (X, T) to the first return map to a subset $Y \subset X$.

Proposition: Let $(X, \mathcal{B}, T, \mu)$ be an ergodic dynamical system and $Y \in \mathcal{B}$ a set with $\mu(Y) > 0$. Let $T_Y = T^{\tau_Y}$ be the first return map to Y.

If μ is T-invariant, then $\nu(A) := \frac{1}{\mu(Y)}\mu(A \cap Y)$ is T_Y-invariant.

Conversely, if ν is T_Y-invariant, and

$$\Lambda := \int_Y \tau(y) d\nu < \infty,$$

then

$$\mu(A) = \frac{1}{\Lambda} \sum_{j=1}^{\infty} \nu(T^{-j}(A) \cap \{y \in Y : \tau(y) \geq j\})$$

is a T-invariant probability measure. Moreover μ is ergodic for T if and only if ν is ergodic for T_Y.

Induced Tranformations

Proof: Let $A \subset Y$ be measurable. We can write $T_Y^{-1}(A)$ as disjoint union $T_Y^{-1}(A) = \sqcup_{k \geq 1} Y_k \cap T^{-k}(A)$, where $Y_k = \{y \in Y : \tau(y) = k\}$. Using the notation of the previous proof, we compute

$$\begin{aligned}
\mu(A) &= \mu(L_0 \cap A) \\
&= \mu(L_1 \cap T^{-1}(A)) + \mu(Y_1 \cap T^{-1}(A)) \\
&= \mu(L_2 \cap T^{-2}(A)) + \mu(Y_2 \cap T^{-2}(A)) + \mu(Y_1 \cap T^{-1}(A)) \\
&= \qquad \vdots \qquad\qquad \vdots \\
&= \sum_{j \geq 1} \mu(Y_j \cap T^{-j}(A)) = \mu(T_Y^{-1}(A)).
\end{aligned}$$

After scaling by $1/\mu(Y)$, we get $\nu(A) = \nu(T_Y^{-1}(A))$.

Induced Tranformations

Conversely, note that

$$\begin{aligned}
\mu(X) &= \frac{1}{\Lambda}\sum_{j=1}^{\infty}\nu(\{y \in Y : \tau(y) \geq 1\}) \\
&= \frac{1}{\Lambda}\sum_{j=1}^{\infty} j\nu(\{y \in Y : \tau(y) = j\}) \\
&= \frac{1}{\Lambda}\int_Y \tau \, d\nu = 1.
\end{aligned}$$

Induced Tranformations

For the invariance, we compute

$$
\begin{aligned}
&\mu(T^{-1}(A)) \\
=&\frac{1}{\Lambda}\sum_{j=1}^{\infty}\nu(T^{-(j+1)}(A)\cap\{\tau(y)\geq j\}) \\
=&\frac{1}{\Lambda}\sum_{j=1}^{\infty}\nu(T^{-(j+1)}(A)\cap\{\tau(y)\geq j+1\})+\nu(T^{-(j+1)}(A)\cap\{\tau(y)=j\}) \\
=&\frac{1}{\Lambda}\sum_{j=1}^{\infty}\left(\nu(T^{-j}(A)\cap\{\tau(y)\geq j\})+\nu(T^{-j}(T^{-1}(A))\cap\{\tau(y)=j\})\right) \\
&\qquad -\frac{1}{\Lambda}\nu(T^{-1}(A)\cap\{\tau(y)\geq 1\}) \\
=&\mu(A)+\frac{1}{\Lambda}\left(\nu(T_Y^{-1}(T^{-1}(A))-\nu(T^{-1}(A))\right)=\mu(A),
\end{aligned}
$$

where the last equality is by T_Y-invariance of ν.

Induced Tranformations

Now for ergodicity, first assume that μ is ergodic and $A \subset Y$ is T_Y-invariant. Then

$$B = \cup_{j=0}^{\infty} T^{-j}(A)$$

is T-invariant, so $\mu(B)$ or $\mu(B^c) = 0$.
If $\mu(B) = 0$ then $\nu(A) = \frac{1}{\mu(Y)}\mu(B \cap Y) = 0$.
If $\mu(B^c) = 0$, then $\nu(A^c) = \frac{1}{\mu(Y)}\mu(B^c \cap Y) = 0$.

Finally, if ν is ergodic and B is T-invariant, then $A := B \cap Y$ is T_Y-invariant, and therefore $\nu(A)$ or $\nu(A^c) = 0$. Suppose the first.

As $\nu|_Y = \frac{1}{\mu(Y)}\mu|_Y$, we have $\mu(A) = 0$. Since T is non-singular, it follows that $\mu(\cup_{j\geq 0} T^{-j}(A))$. But μ-a.e., $x \in B$ belongs to $\cup_{j\geq 0} T^{-j}(A)$, so also $\mu(B) = 0$.

The case $(A^c) = 0$ goes likewise.

Induced Tranformations

As an illustration, we take the quadratic map $T(x) = 4x(1-x)$. It is not uniformly expanding, so we cannot apply the Folklore Theorem to find an absolutely continuous probability measure μ.

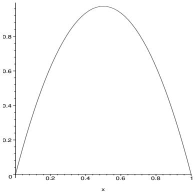

Figure: The quadratic map $T : x \mapsto 4x(1-x)$

Therefore we take $Y = [1-p, p]$ for the fixed point $p = \frac{3}{4}$ of T. and consider the first return map $T_Y : Y \to Y$. Note that the critical point $c = \frac{1}{2}$ satisfies $T(c) = 1$ and $T^2(c) = 0$ is fixed under f. This is essential for T_Y to be uniformly expanding.

Induced Tranformations

Without proofs, we mention the properties of T_Y:

- T_Y is defined for Lebesgue-a.e. $y \in Y$.
- If $y \in Y$ has return time $\tau(y) = n$, then there is a neighborhood U_x of x such that $T_Y : U_x \to Y^\circ$ is a C^∞ diffeomorphism and $|T_Y'| \geq 2$.
- T_Y has infinitely many branches (so it is not piecewise C^2 in the strict sense), and T_Y' is not bounded. However, there is a constant C such that

$$\frac{|T_Y''(y)|}{|T_Y'(y)|^2} \leq C \qquad \text{wherever defined.}$$

- The Lebesgue measure of $\{y \in Y : \tau(y) = n\}$ is exponentially small in n.

These conditions are sufficient to get the conclusion of the Folklore Theorem, so we have an T_Y-invariant measure ν and in fact, its density $\frac{d\nu}{dx}$ is bounded and bounded away from zero.

Induced Tranformations

This means that

$$\nu(\{y \in Y : \tau(y) = n\}) \textit{ is exponentially small in } n$$

as well, so that the normalizing constant $\Lambda < \infty$. Hence, we conclude that f preserves an ergodic absolutely continuous measure μ, satisfying the formula of the previous proposition.

For the above example, it is not essential that T is a quadratic map; any C^2 unimodal map $T : [0,1] \to [0,1]$ with $T^2(c) = 0$ fixed and $T''(c) \neq 0$ can be treated in the same way. For the quadratic map, however, the density of μ is known precisely:

$$\frac{d\mu}{dx} = \frac{1}{\pi\sqrt{x(1-x)}}.$$