The Koopman Operator

Given $(X, \mathcal{B}, \mu, T)$, we can take the space of complex-valued square-integrable observables $L^2(\mu)$. This is a Hilbert space, equipped with inner product

$$\langle f, g \rangle = \int_X f(x) \cdot \overline{g(x)} \, d\mu.$$

The Koopman operator is defined as

$$U_T : L^2(\mu) \to L^2(\mu), \qquad U_T f = f \circ T.$$

Exercise 8.1: Show that the Koopman operator is linear and positive.

The Koopman Operator

By T-invariance of μ, it is a unitary operator (it preserves the inner product). Indeed

$$\begin{aligned}\langle U_T f, U_T g\rangle &= \int_X f\circ T(x)\cdot\overline{g\circ T(x)}\,d\mu \\ &= \int_X (f\cdot\overline{g})\circ T(x)\,d\mu = \int_X f\cdot\overline{g}\,d\mu = \langle f, g\rangle,\end{aligned}$$

and therefore $U_T^* U_T = U_T U_T^* = I$.

Remark: This has several consequences, common to all unitary operators. The spectrum $\sigma(U_T)$ of U_T is a closed subset of the unit circle, and it is in fact a group under multiplication.

The Transfer Operator

Definition: The Perron-Frobenius or transfer operator of a transformation $T : X \to X$ is the dual of the Koopman operator:

$$\int_X P_T f \cdot g \; d\mu = \int_X f \cdot U_T g \; d\mu = \int_X f \cdot g \circ T \; d\mu.$$

Note that, although U_T is independent of the measure, P_T is not. Often it will be important to specify the measure explicitly, and this measure need not be invariant.

Exercise 8.2: Show that the Perron-Frobenius operator has the following properties:

1. P_T is linear;
2. P_T is positive: $f \geq 0$ implies $P_T f \geq 0$.
3. $\int P_T f \; d\mu = \int f \; d\mu$.
4. $P_{T^k} = (P_T)^k$.

The Transfer Operator

Lemma: Let $T : [0,1] \to [0,1]$ be a piecewise C^1 interval map. Then the Perron-Frobenius operator P_T w.r.t. Lebesgue measure λ has the pointwise formula

$$P_T f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}.$$

Specifically, a fixed point of P is an invariant density for T.

Proof Let $0 = a_0 < a_1 < \cdots < a_N = 1$ be such that T is C^1 monotone on each (a_{i-1}, a_i). Let $y_i = T^{-1}(x) \cap (a_{i-1}, a_i)$.

The Transfer Operator

We obtain

$$
\begin{aligned}
(P_T f)(x) \underbrace{=}_{\text{Fund. Th. Calc}} & \frac{d}{dx}\int_0^x P_T f(s)\, ds = \frac{d}{dx}\int_0^1 (P_T f)(s)\cdot 1_{[0,x]}(s) ds \\
\underbrace{=}_{\text{def. } P_T} & \frac{d}{dx}\int_0^1 f \cdot 1_{[0,x]} \circ T(s)\, ds \underbrace{=}_{s=T(u)} \frac{d}{dx}\int_{T^{-1}[0,x]} f(u)\, du \\
= & \sum_{\substack{T|_{(a_{i-1},a_i)} \text{ increasing} \\ x\in T((a_{i-1},a_i))}} \frac{d}{dx}\int_{a_{i-1}}^{y_i(x)} f(u)\, du \\
& + \sum_{\substack{T|_{(a_{i-1},a_i)} \text{ decreasing} \\ x\in T((a_{i-1},a_i))}} \frac{d}{dx}\int_{y_i(x)}^{a_i} f(u)\, du \\
& + \sum_{T((a_{i-1},a_i))\subset[0,x]} \frac{d}{dx}\int_{a_{i-1}}^{a_i} f(u)\, du
\end{aligned}
$$

The Transfer Operator

Continuing this:

$$\begin{aligned}(P_T f)(x) &= \sum_{\substack{T|_{(a_{i-1},a_i)} \text{ increasing} \\ x\in T((a_{i-1},a_i))}} \frac{f(y_i)}{T'(y_i)} + \sum_{\substack{T|_{(a_{i-1},a_i)} \text{ decreasing} \\ x\in T((a_{i-1},a_i))}} -\frac{f(y_i)}{T'(y_i)} + 0 \\ &= \sum_i \frac{f(y_i)}{|T'(y_i)|},\end{aligned}$$

as required.

The Transfer Operator

There is also a Perron-Frobenius operator with respect to a measure $\mu \ll$ Lebesgue, instead of Lebesgue measure itself:

Lemma: If $d\mu = hdx$, then the operator

$$P_{T,\mu}f = \frac{P_T(f \cdot h)}{h}$$

acts as the Perron-Frobenius operator on $(X, \mathcal{B}, T, \mu)$.

Viewed differently, if a function $h \geq 0$ is fixed by P_T (w.r.t. Lebesgue) then $d\mu = h\ dx$ is an invariant measure.

The Transfer Operator

Proof of the Lemma: Let A be μ-measurable and $f \in L^1(\mu)$. Then

$$
\begin{aligned}
\int_A P_{T,\mu} f \; d\mu &= \int_A \frac{P_T(f \cdot h)}{h} \, h \, dx \\
&= \int_A P_T(f \cdot h) \, dx \\
&= \int_X P_T(f \cdot h) \, 1_A \, d\mu = \int_X f \cdot h \cdot (1_A \circ T) \, dx \\
&= \int_{T^{-1}A} f \cdot h \, dx = \int_{T^{-1}A} f \; d\mu.
\end{aligned}
$$

Because A is arbitrary, the lemma is proved.

Mixing

Definition: A probability measure preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is mixing (or strong mixing) if

$$\mu(T^{-n}(A) \cap B) \to \mu(A)\mu(B) \text{ as } n \to \infty$$

for every $A, B \in \mathcal{B}$.

This says that the "events" A and B are asymptotically independent.

Exercise: Show that Lebesgue measure μ is mixing for the doubling map. In fact, the n-th correlation coefficient

$$\mathrm{Cor}_n(A, B) := \mu(T^{-n}(A) \cap B) - \mu(A)\mu(B) = 0$$

for every $n \geq 1$.

Mixing

Proposition: A probability preserving dynamical system $(X, \mathcal{B}, T, \mu)$ is mixing if and only if

$$\int_X f \circ T^n(x) \cdot \overline{g(x)}\, d\mu \to \int_X f(x)\, d\mu \cdot \int_X \overline{g(x)}\, d\mu \text{ as } n \to \infty$$

for all $f, g \in L^2(\mu)$, or written in the notation of the Koopman operator $U_T f = f \circ T$:

$$\langle U_T^n f, g \rangle \to \langle f, 1 \rangle \langle 1, g \rangle \text{ as } n \to \infty.$$

Proof: The "if"-direction follows by taking indicator functions $f = 1_A$ and $g = 1_B$. For the "only if"-direction, general $f, g \in L^2(\mu)$ can be approximated by linear combinations of indicator functions.

Exponential Mixing

Assume without loss of generality that $\int g \, d\mu = \overline{\langle 1, g \rangle} = 0$. Then mixing means that

$$\langle U_T^n f, g \rangle = \langle f, P_T^n g \rangle \to 0 \text{ as } n \to \infty.$$

If the operator norm $\|P_T\|$ restricted to $\{g \in L^2(\mu) : \int g \, d\mu = 0\}$ is strictly less than 1, then

$$\mathrm{Cor}_n(f, g) = \langle f, P_T^n g \rangle \leq \|f\| \, \|g\| \, \|P_T\|^n \to 0 \quad \text{exponentially fast.}$$

Thus μ is exponentially mixing: the "events" f and g are exponentially independent.

This motivates the spectral analysis of the transfer operator.

Exponential Mixing

Definition: An operator P is called quasi-compact if $\lambda = 1$ is an eigenvalue of multiplicity one, and the rest of the spectrum (i.e., all $\lambda \in \mathbb{C}$ such that $(P - \lambda I)^{-1}$ is not a wel-defined bounded operator), lies in a disk $\{|\lambda| \leq \sigma\}$ for some $\sigma < 1$.

- Multiplicity one means that the space of eigenfunctions satisfying $Pf = \lambda f$ (for $\lambda = 1$) is one-dimensional.
- This σ is called the essential spectral radius. The actual spectral radius is 1.
- Perron-Frobenius operators are seldom quasi-compact on $L^2(\mu)$. The (non-trivial) task is to find a Banach space $\mathcal{B}$ restricted to which P is quasi-compact.

Exponential Mixing

The Lasota-Yorke (or Doeblin-Fortet or Tulcea-Ionescu-Marinescu) inequality holds if there are two Banach spaces $(\mathcal{B}_s, \| \ \|_s)$ and $(\mathcal{B}_w, \| \ \|_w)$ (for strong and weak) and $\sigma \in (0,1)$, $L \geq 1$ such that

- $B_s \subset B_w$ and the unit ball $\{f \in B_s : \|f\|_s \leq 1\}$ is a compact subset of B_w.
- $\|Pf\|_s \leq \sigma \|f\|_s + L\|f\|_w$ for all $f \in B_s$.
- $\|Pf\|_w \leq \|f\|_w$ for all $f \in B_w$.

Under these conditions, $P : B_s \to B_s$ is quasi-compact, with essential spectral radius $\leq \sigma$.

Theorem: If the Perron-Frobenius operator for $T : [0,1] \to [0,1]$ is quasi-compact with $B_s \subset L^1(m)$ for Lebesgue measure m, then there is an absolutely continuous invariant measure μ which is exponentially mixing.

Exponential Mixing

Exercise 8.3: Show that $\|P_T f\|_{L^1} \leq \|f\|_{L^1}$.

Exercise 8.4: Suppose we have two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ satisfying: there are $\sigma \in (0,1)$ and $L > 0$ such that:

$$a_{n+1} \leq a_n \quad \text{and} \quad b_n \leq \sigma b_n + L a_n.$$

Show that $(b_n)_{n\in\mathbb{N}}$ is bounded. Show that $\limsup_n b_n \leq L/(1-\sigma)$.

The Lasota-Yorke inequality got reinvented several times (Andrej Lasota and James Yorke being the most recent), and they applied this to expanding interval maps with specific choices of Banach spaces.

Exponential Mixing

Definition: Let $g : [a, b] \to \mathbb{R}$. The *variation* of g is defined to be

$$\mathrm{Var}_{[a,b]}\, g = \sup \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|,$$

where the supremum runs over all finite partitions generated by points $a = x_0 < x_1 < \cdots < x_n = b$.

Note that Var is a seminorm, i.e., $\mathrm{Var}(f) = 0 \not\Rightarrow f \equiv 0$. In fact, $\mathrm{Var}\, f = \mathrm{Var}(f + C)$ for every constant C. However,

$$BV := \{f \in L^1 : \mathrm{Var}(f) < \infty\} \text{ with norm } \|f\|_{BV} = \mathrm{Var}(f) + \int |f| dx$$

is a Banach space. Its unit ball is compactly embedded in L^1.

Exponential Mixing

Lasota-Yorke Theorem: If $T : [0,1] \to [0,1]$ is a piecewise C^2 map with $\inf_{x \in [0,1]} |T'(x)| > 1$, then T preserves an invariant measure $\mu \ll m =$ Lebesgue, and the density $h = \frac{d\mu}{dx}$ is of bounded variation. Also μ is exponentially mixing.

- Compared to the Folklore Theorem, there is an important difference that the branches need not be onto.
- For instant, β-transformations:

$$T_\beta : [0,1] \to [0,1], \qquad T_\beta(x) = \beta x \bmod 1$$

have an absolutely continuous measures for all $\beta > 1$, not just the integers $\beta \geq 2$.

Exponential Mixing

The technical details are in the notes, but the key-steps are:

- Take an iterate T^n such that $\inf_x |(T^n)'(x)| > 2$, so $\sigma := 2/\inf_x |(T^n)'(x)| < 1$.
- For $B_s = BV$ and $B_w = L^1(m)$ the Lasota-Yorke ineq. holds:

$$\mathrm{Var}_{[0,1]} P_{T,m} g \leq \sigma\, \mathrm{Var}_{[0,1]} g + L \|g\|_{L^1(m)}.$$

- The limit μ of Césaro means $\frac{1}{n}\sum_{k=0}^{n-1} m \circ T^{-k}$ is invariant.
- The invariant density $h = \frac{d\mu}{dx} = \lim_n P^n_{T,m} 1 \in BV$.
- The eigenspace of $P_{T,m}$ for $\lambda = 1$ is spanned by h.
- For all $f \in L^1(m)$ and $g \in BV$:

$$\begin{aligned} \mathrm{Cor}_n(f,g) &= \int f \cdot P^n_{T,m}\, g\, h\, dx - \int f\, d\mu \int g\, d\mu \\ \text{because } P^n_{T,m} h = h \quad &= \int f \cdot P_{T,m}\, (g - \int g\, d\mu)\, h\, dx \\ &\leq \|f\|_{L^1} \|g\|_{BV} \|P^n|_{h^\perp}\| \\ &\leq \|f\|_{L^1}\, \|g\|_{BV}\, \sigma^n. \end{aligned}$$