Bernoulli Shifts

Beroulli trials in probability refer to successive coinflips or roles of the die. They are modeled as a shift-space over an alphabet $\mathcal{A} = \{1, \ldots, N\}$:

$$\Sigma = \mathcal{A}^{\mathbb{Z}} \text{ (two-sided) or } \Sigma = \mathcal{A}^{\mathbb{N}} \text{(one-sided)}$$

with a probability vector

$$p = (p_1, \ldots p_N) \qquad p_i \in [0,1], \ \sum_{i=1}^{N} p_i = 1.$$

The corresponding Bernoulli measure μ_p gives cylinder sets

$$Z_{[k+1,k+n]}(a) = \{x \in \Sigma : x_{k+1} \ldots x_{k+n} = a_1 \ldots a_n\}$$

the mass

$$\mu_p(\, Z_{[k+1,k+N]}(a)\,) = \prod_{j=1}^{n} p_{a_i}.$$

Bernoulli Shifts

The Kolmogorov Extension Theorem allows one to extend μ_p to every set in the Borel σ-algebra $\mathcal{B}$ of Σ. (Note that the cylinder sets form a basis of the topology.)

The measure space $(\Sigma, \mathcal{B}, \mu_p)$ can be made into a measure preserving dynamical systems by taking the left-shift

$$\sigma(\ \ldots x_{-2}x_{-1} \ . \ x_0x_1x_2 \ldots \ ) = \ \ldots x_{-2}x_{-1}x_0 \ . \ x_1x_2 \ldots$$

Exercise: Show that the set of periodic points of $(\Sigma, \mathcal{B}, \mu_p; \sigma)$ has measure zero if and only if $p_i < 1$ for all i.

Exercise: Show that the set of points in $(\Sigma, \mathcal{B}, \mu_p; \sigma)$ with a dense orbit has measure one if and only if $p_i > 0$ for all i.

Isomorphic Systems

Definition: Two measure preserving dynamical systems $(X, \mathcal{B}, T, \mu)$ and $(Y, \mathcal{C}, S, \nu)$ are called isomorphic if there are $X' \in \mathcal{B}$, $Y' \in \mathcal{C}$ and $\phi : Y' \to X'$ such that

- $\mu(X') = 1$, $\nu(Y') = 1$;
- $\phi : Y' \to X'$ is a bi-measurable bijection;
- ϕ is measure preserving: $\nu(\phi^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$.
- $\phi \circ S = T \circ \phi$.

That is, the below diagram commutes, and $\phi : Y \to X$ is one-to-one almost everywhere.

$$\begin{array}{ccc} (Y, \mathcal{C}, \nu) & \xrightarrow{S} & (Y, \mathcal{C}, \nu) \\ \phi \downarrow & & \downarrow \phi \\ (X, \mathcal{B}, \mu) & \xrightarrow{T} & (X, \mathcal{B}, \mu) \end{array}$$

Isomorphic Systems

Example: The doubling map

$$T : \mathbb{S}^1 \to \mathbb{S}^1, \qquad T(x) = 2x \bmod 1$$

with Lebesgue measure is isomorphic to the one-sided $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift $(\Sigma, \mathcal{B}, \sigma, \mu)$. The (inverse of the) isomorphism is the coding map $\phi^{-1} : X' \to \Sigma'$:

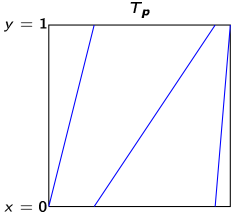
$$\phi^{-1}(x)_k = \begin{cases} 1 & \text{if } T^k(x) \in [0, \frac{1}{2}), \\ 2 & \text{if } T^k(x) \in (\frac{1}{2}, 1]. \end{cases}$$

Here $X' = [0, 1] \setminus \{\text{dyadic rationals in } (0, 1)\}$ because these dyadic rationals map to $\frac{1}{2}$ under some iterate of T, and at $\frac{1}{2}$ the coding map is not well-defined. Note that

$$\Sigma' = \{0, 1\}^{\mathbb{N}} \setminus \{v10^\infty, v01^\infty : v \text{ is a finite word in } \{0, 1\}\}.$$

Isomorphic Systems

Example: Let $(p_1, \dots, p_N)$ be some probability vector with all $p_i > 0$. Then the one-sided $(p_1, \dots, p_N)$-Bernoulli shift is isomorphic to $([0,1], \mathcal{B}, T, Leb)$ where $T : [0,1] \to [0,1]$ has N linear branches of slope $1/p_i$.



The map T_p for

$p = (\frac{1}{4}, \frac{2}{3}, \frac{1}{12})$

The one-sided $(p_1, \dots, p_N)$-Bernoulli shift is also isomorphic to

$$([0,1], \mathcal{B}, S, \nu) \text{ where } S(x) = Nx \bmod 1.$$

But here ν is another measure that gives $[\frac{i-1}{N}, \frac{i}{N}]$ the mass p_i, and $[\frac{i-1}{N} + \frac{j-1}{N^2}, \frac{i-1}{N} + \frac{j}{N^2}]$ the mass $p_i p_j$, etc.

Ornstein Theorem

The entropy of the Bernoulli system is defined as:

$$h(p) := -\sum_{i=1}^{N} p_i \log p_i \qquad \text{Convention: } 0 \log 0 = 0.$$

It is preserved under isomorphism, so

Two isomorphic Bernoulli system have the same entopy.

Theorem of Ornstein: Two 2-sided Bernoulli systems are isomorphic if and only if they have the same entropy.

Exercise: The 2-sided $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ shift is isomorphic to the 2-sided $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2})$-shift, but to no other 2-sided shift on ≤ 4 symbols.

Natural Extensions

Invertible systems cannot be isomorphic to non-invertible systems. So

$$(\{0,1\}^{\mathbb{Z}}, \mathcal{B} : \sigma) \text{ and } (\{0,1\}^{\mathbb{N}}, \mathcal{B} : \sigma)$$

are not isomorphic.

But there is a construction to make a non-invertible system invertible, namely by passing to the natural extension.

Natural Extensions

Definition: Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving dynamical system. A system $(Y, \mathcal{C}, S, \nu)$ is a natural extension of $(X, \mathcal{B}, \mu, T)$ if there are $X' \in \mathcal{B}$, $Y' \in \mathcal{C}$ and $\phi : Y' \to X'$ such that

- $\mu(X') = 1$, $\nu(Y') = 1$;
- $S : Y' \to Y'$ is invertible;
- $\phi : Y' \to X'$ is a measurable surjection;
- ϕ is measure preserving: $\nu(\phi^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$;
- $\phi \circ S = T \circ \phi$.

Any two natural extensions can be shown to be isomorphic, so it makes sense to speak of the natural extension.

Natural Extensions

Sometimes natural extensions have explicit formulas. For example the baker map $B : [0,1]^2 \to [0,1]^2$,

$$B(x,y) = \begin{cases} (2x, \frac{y}{2}) & \text{if } x < \frac{1}{2}; \\ (2x-1, \frac{y+1}{2}) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

preserving two-dimensional Lebesgue measure is the natural extension of the doubling map via the factor map $\phi(x,y) = x$.

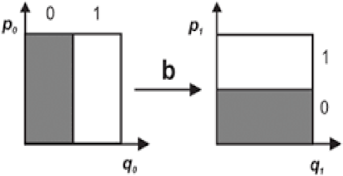


FIG. 1: The baker map propagates a pair of vertical rectangles onto a pair of horizontal rectangles. Inside each separate region the evolution is linear.

Natural Extensions

There is also a general construction: Set

$$Y = \{(x_i)_{i\geq 0} : T(x_{i+1}) = x_i \in X \text{ for all } i \geq 0\}$$

with $S((x_0, x_1, \dots)) = (T(x_0), x_0, x_1, \dots)$. Then S is invertible (with the left shift $\sigma = S^{-1}$) and

$$\nu(A_0, A_1, A_2, \dots) = \inf_i \mu(A_i) \quad \text{for } (A_0, A_1, A_2 \dots) \subset S,$$

is S-invariant.

The factor map $\phi(x_0, x_1, x_2, \dots) := x_0$ satisfies $T \circ \phi = \phi \circ S$.

Also ϕ is measure preserving because, for each $A \in \mathcal{B}$,

$$\phi^{-1}(A) = (A, T^{-1}(A), T^{-2}(A), T^{-3}(A), \dots)$$

and clearly $\nu(A, T^{-1}(A), T^{-2}(A), T^{-3}(A), \dots) = \mu(A)$ because $\mu(T^{-i}(A)) = \mu(A)$ for every i by T-invariance of μ.

The Bernoulli Property

Definition: Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving dynamical system.

1. If T is invertible, then the system is called Bernoulli if it is isomorphic to a 2-sided Bernoulli shift.
2. If T is non-invertible, then the system is called one-sided Bernoulli if it is isomorphic to a 1-sided Bernoulli shift.
3. If T is non-invertible, then the system is called Bernoulli if its natural extension is isomorphic to a 2-sided Bernoulli shift.

The Bernoulli Property

The third Bernoulli property is quite general, even though the isomorphism ϕ may be very difficult to find explicitly. Thus, proving that a system is not Bernoulli can be hard.

Expanding circle maps $T : \mathbb{S}^1 \to \mathbb{S}^1$ that satisfy the conditions of the Folklore Theorem are also Bernoulli, i.e., have a Bernoulli natural extension (proven by Ledrappier).

Being one-sided Bernoulli, on the other hand, is quite special. If T is piecewise C^2 but not piecewise linear, then it has to be C^2-conjugate to a piecewise linear expanding map to be one-sided Bernoulli.

Bernoulli (of any of the above forms) implies mixing. So non-mixing systems (e.g. circle rotations) are not Bernoulli.