Notes on Ergodic Theory.

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Abstract

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1 Notation

Throughout, (X, d) will be a metric space, possibly compact, and $T : X \to X$ will be a (piecewise) continuous map. The combination (X, T) defines dynamical systems by means of iteration. The **orbit** of a point $x \in X$ is the set

$$\mathrm{orb}(x) = \{x, T(x), T \circ T(x), \dots, \underbrace{T \circ \dots \circ T}_{n \text{ times}}(x) =: T^n(x), \cdots\} = \{T^n(x) : n \geqslant 0\},$$

and if T is invertible, then $\mathrm{orb}(x) = \{T^n(x) : n \in \mathbb{Z}\}$ where the negative iterates are defined as $T^{-n} = (T^{inv})^n$. In other words, we consider $n \in \mathbb{N}$ (or $n \in \mathbb{Z}$) as discrete time, and $T^n(x)$ is the position the point x takes at time n.

Definition 1. *We call x a* **fixed point** *if $T(x) = x$;* **periodic** *if there is $n \geqslant 1$ such that $T^n(x) = x$;* **recurrent** *if $x \in \overline{\mathrm{orb}(x)}$.*

In general *chaotic* dynamical systems most orbits are more complicated than periodic (or quasi-periodic as the irrational rotation R_α discussed below). The behaviour of such orbits is hard to predict. Ergodic Theory is meant to help in predicting the behaviour of **typical** orbits, where typical means: almost all points x for some (invariant) measure μ.

To define measures properly, we need a σ-algebra $\mathcal{B}$ of "measurable" subsets. σ-algebra means that the collection $\mathcal{B}$ is closed under taking complements, countable unions and countable intersections, and also that $\varnothing, X \in \mathcal{B}$. Then a measure μ is a function

$\mu : \mathcal{B} \to \mathbb{R}^+$ that is countably subadditive: $\mu(\cup_i A_i) \leqslant \sum_i \mu(A)_i$ (with equality if the sets A_i are pairwise disjoint).

Example: For a subset $A \subset X$, define

$$\nu(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_A \circ T^i x,$$

for the **indicator function** 1_A, assuming for the moment that this limit exists. We call this the visit frequency of x to the set A. We can compute

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_A \circ T^i x &= \lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=0}^{n-1} 1_A \circ T^{i+1} x + 1_A x - 1_A(T^n x) \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=0}^{n-1} 1_{T^{-1}A} \circ T^i x + 1_A x - 1_A(T^n x) \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{T^{-1}A} \circ T^i x = \nu(T^{-1}(A))
\end{aligned}$$

That is, visit frequency measures, when well-defined, are **invariant** under the map. This allows us to use invariant measure to make statistical predictions of what orbit do "on average".

Let $\mathcal{B}_0$ be the collection of subsets $A \in \mathcal{B}$ such that $\mu(A) = 0$, that is: $\mathcal{B}_0$ are the **null-sets** of μ. We say that an event happens almost surely (a.s.) or μ-almost everywhere (μ-a.e.) if it is true for all $x \in X \setminus A$ for some $A \in \mathcal{B}_0$.

A measure μ on $(X, T, \mathcal{B})$ is called

- non-singular if $A \in \mathcal{B}_0$ implies $T^{-1}(A) \in \mathcal{B}_0$.
- non-atomic if $\mu(\{x\}) = 0$ for every $x \in X$
- T-invariant if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.
- finite if $\mu(X) < \infty$. In this case we can always rescale μ so that $\mu(X) = 1$, *i.e.,* μ is a probability measure.
- σ-finite if there is a countable collection X_i such that $X = \cup_i X_i$ and $\mu(X_i) \leqslant 1$ for all i. In principle, finite measures are also σ-finite, but we would like to reserve the term σ-finite only for infinite measures (*i.e.,* $\mu(X) = \infty$).

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{for matrix} \qquad M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

2 What are its invariant measures?

Note that T is a bijection of $\mathbb{R}^2$, with 0 as single fixed point. Therefore the Dirac measure δ_0 is T-invariant. However, also Lebesgue measure m is invariant because (using coordinate transformation $x = T^{-1}(y)$)

$$m(T^{-1}A) = \int_{T^{-1}A} dm(x) = \int_A \det(M^{-1}) dm(y) = \int_A \frac{1}{\det(M)} dm(y) = m(A)$$

because $\det(M) = 1$. This is a general fact: If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection with Jacobian $J = |\det(DT)| = 1$, then Lebesgue measure is preserved. However, Lebesgue measure is not a probability measure (it is σ-finite). In the above case of the integer matrix with determinant 1, T preserves (and is a bijection) on $\mathbb{Z}^2$. Therefore we can factor out over $\mathbb{Z}^2$ and obtain a map on the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$:

$$\begin{aligned} T \ :& \ \mathbb{T}^2 \to \mathbb{T}^2 \\ & \begin{pmatrix} x \\ y \end{pmatrix} \mapsto M \begin{pmatrix} x \\ y \end{pmatrix} \pmod 1 \end{aligned}$$

This map is called Arnol'd's cat-map, and it preserves Lebesgue measure, which on $\mathbb{T}^2$ is a probability measure.

A special case of the above is:

Proposition 1. *If $T : U \subset \mathbb{R}^n \to U$ is an isometry (or piecewise isometric bijection), then T preserves Lebesgue measure.*

Let $\mathcal{M}(X, T)$ denote the set of T-invariant Borel[1] probability measures. In general, there are always invariant measures.

Theorem 1 (Krylov-Bogol'ubov)**.** *If $T : X \to X$ is a continuous map on a nonempty compact metric space X, then $\mathcal{M}(T) \neq \varnothing$.*

Proof. Let ν be any probability measure and define Cesaro means:

$$\nu_n(A) = \frac{1}{n} \sum_{j=0}^{n-1} \nu(T^{-j}A),$$

these are all probability measures. The collection of probability measures on a compact metric space is known to be compact in the weak* topology, *i.e.,* there is limit probability measure μ and a subsequence $(n_i)_{i \in \mathbb{N}}$ such that for every continuous function $\psi : X \to \mathbb{R}$:

$$\int_X \psi \, d\nu_{n_i} \to \int \psi \, d\mu \text{ as } i \to \infty. \tag{1}$$

[1]that is, sets in the σ-algebra of sets generated by the open subsets of X.

On a metric space, we can, for any $\varepsilon > 0$ and closed set A, find a continuous function $\psi_A : X \to [0,1]$ such that $\psi_A(x) = 1$ if $x \in A$ and $\mu(A) \leqslant \int_X \psi_A d\mu \leqslant \mu(A) + \varepsilon$ and similarly $\mu(T^{-1}A) \leqslant \int_X \psi_A \circ T \, d\mu \leqslant \mu(T^{-1}A) + \varepsilon$. Now

$$\begin{aligned}
|\mu(T^{-1}(A)) - \mu(A)| &\leqslant \left| \int \psi_A \circ T \ d\mu - \int \psi_A \ d\mu \right| + 2\varepsilon \\
&= \lim_{i\to\infty} \left| \int \psi_A \circ T \ d\nu_{n_i} - \int \psi_A \ d\nu_{n_i} \right| + 2\varepsilon \\
&= \lim_{i\to\infty} \frac{1}{n_i} \left| \sum_{j=0}^{n_i-1} \left(\int \psi_A \circ T^{-(j+1)} \ d\nu - \int \psi_A \circ T^{-j} \ d\nu \right) \right| + 2\varepsilon \\
&\leqslant \lim_{i\to\infty} \frac{1}{n_i} \left| \int \psi_A \circ T^{-n_i} \ d\nu - \int \psi_A \ d\nu \right| + 2\varepsilon \\
&\leqslant \lim_{i\to\infty} \frac{1}{n_i} 2\|\psi_A\|_\infty + 2\varepsilon = 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, and because the closed sets form a generator of the Borel sets, we find that $\mu(T^{-1}(A)) = \mu(A)$ as required. □

3 Ergodicity and unique ergodicity

Definition 2. *A measure is called* **ergodic** *if* $T^{-1}(A) = A \pmod{\mu}$ *for some* $A \in \mathcal{B}$ *implies that* $\mu(A) = 0$ *or* $\mu(A^c) = 0$.

Proposition 2. *The following are equivalent:*

(i) μ *is ergodic;*

(ii) If $\psi \in L^1(\mu)$ *is* T*-invariant,* i.e., $\psi \circ T = \psi$ μ*-a.e., then* ψ *is constant* μ*-a.e.*

Proof. (i) ⇒ (ii): Let $\psi : X \to \mathbb{R}$ be T-invariant μ-a.e., but not constant. Thus there exists $a \in \mathbb{R}$ such that $A := \psi^{-1}((-\infty, a])$ and $A^c = \psi^{-1}((a, \infty))$ both have positive measure. By T-invariance, $T^{-1}A = A \pmod{\mu}$, and we have a contradiction to ergodicity.
(ii) ⇒ (i): Let A be a set of positive measure such that $T^{-1}A = A$. Let $\psi = 1_A$ be its indicator function; it is T-invariant because A is T-invariant. By (ii), ψ is constant μ-a.e., but as $\psi(x) = 0$ for $x \in A^c$, it follows that $\mu(A^c) = 0$. □

The rotation $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ is defined as $R_\alpha(x) = x + \alpha \pmod 1$.

Theorem 2 (Poincaré)**.** *If* $\alpha \in \mathbb{Q}$*, then every orbit is periodic.*

If $\alpha \notin \mathbb{Q}$, then every orbit is dense in $\mathbb{S}^1$. In fact, for every interval J and every $x \in \mathbb{S}^1$, the visit frequency

$$v(J) := \lim_{n\to\infty} \frac{1}{n} \#\{0 \leqslant i < n : R_\alpha^i(x) \in J\} = |J|.$$

Proof. If $\alpha = \frac{p}{q}$, then clearly

$$R_\alpha^q(x) = x + q\alpha \pmod 1 = x + q\frac{p}{q} \pmod 1 = x + p \pmod 1 = x.$$

Conversely, if $R_\alpha^q(x) = x$, then $x = x + q\alpha \pmod 1$, so $q\alpha = p$ for some integer p, and $\alpha = \frac{p}{q} \in \mathbb{Q}$.

Therefore, if $\alpha \notin \mathbb{Q}$, then x cannot be periodic, so its orbit is infinite. Let $\varepsilon > 0$. Since $\mathbb{S}^1$ is compact, there must be $m < n$ such that $0 < \delta := d(R_\alpha^m(x), R_\alpha^n(x)) < \varepsilon$. Since R_α is an isometry, $|R_\alpha^{k(n-m)}(x) - R_\alpha^{(k+1)(n-m)}(x)| = \delta$ for every $k \in \mathbb{Z}$, and $\{R_\alpha^{k(n-m)}(x) : k \in \mathbb{Z}\}$ is a collection of points such that every two neighbours are exactly δ apart. Since $\varepsilon > \delta$ is arbitrary, this shows that $\mathrm{orb}(x)$ is dense, but we want to prove more.

Let $J_\delta^0 = [R_\alpha^m(x), R_\alpha^n(x))$ and $J_\delta^k = R_\alpha^k(J_\delta)$. Then for $K = \lfloor 1/\delta \rfloor$, $\{J_\delta^k\}_{k=0}^K$ is a cover $\mathbb{S}^1$ of adjacent intervals, each of length δ, and $R_\alpha^{j(n-m)}$ is an isometry from J_δ^i to J_δ^{i+j}. Therefore the visit frequencies

$$\underline{v}_k = \liminf_n \frac{1}{n} \#\{0 \leqslant i < n : R_\alpha^i(x) \in J_\delta^k\}$$

are all the same for $0 \leqslant k \leqslant K$, and together they add up to at most $1 + \frac{1}{K}$. This shows for example that

$$\frac{1}{K+1} \leqslant \underline{v}_k \leqslant \overline{v}_k := \limsup_n \frac{1}{n} \#\{0 \leqslant i < n : R_\alpha^i(x) \in J_\delta^k\} \leqslant \frac{1}{K},$$

and these inequalities are **independent** of the point x. Now an arbitrary interval J can be covered by $\lfloor |J|/\delta \rfloor + 2$ such adjacent J_δ^k, so

$$v(J) \leqslant \left(\frac{|J|}{\delta} + 2\right) \frac{1}{K} \leqslant (|J|(K+1) + 2) \frac{1}{K} \leqslant |J| + \frac{3}{K}.$$

A similar computation gives $v(J) \geqslant |J| - \frac{3}{K}$. Now taking $\varepsilon \to 0$ (hence $\delta \to 0$ and $K \to \infty$), we find that the limit $v(J)$ indeed exists, and is equal to $|J|$. □

Definition 3. *A transformation (X, T) is called* **uniquely ergodic** *if there is exactly one invariant probability measure.*

The above proof shows that Lebesgue measure is the only invariant measure if $\alpha \notin \mathbb{Q}$, so $(\mathbb{S}^1, R_\alpha)$ is uniquely ergodic. However, there is a missing step in the logic, in that we didn't show yet that Lebesgue measure is ergodic. This will be shown in Example 1 and also Theorem 6.

Questions: Does R_α preserve a σ-finite measure? Does R_α preserve a **non-atomic** σ-finite measure?

Lemma 1. *Let X be a compact space. A transformation $(X, \mathcal{B}, \mu, T)$ is uniquely ergodic if and only if, for every continuous function, the Birkhoff averages $\frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i(x)$ converge* **uniformly** *to a constant function.*

Remark 1. *Every continuous map on a compact space has an invariant measure, as we showed in Theorem 1. Theorem 6 later on shows that if there is only one invariant measure, it has to be ergodic as well.*

Proof. If μ and ν were two different ergodic measures, then we can find a continuous function $f : X \to \mathbb{R}$ such that $\int f d\mu \neq \int f d\nu$. Using the Ergodic Theorem for both measures (with their own typical points x and y), we see that

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) = \int f d\mu \neq \int f d\nu = \lim_n \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(y),$$

so there is not even convergence to a constant function.

Conversely, we know by the Ergodic Theorem that $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) = \int f d\mu$ is constant μ-a.e. But if the convergence is not uniform, then there are sequences $(x_i), (y_i) \subset X$ and $(m_i), (n_i) \subset \mathbb{N}$, such that $\lim_i \frac{1}{m_i}\sum_{k=0}^{m_i-1} f \circ T^k(x) := A \neq B =: \lim_i \frac{1}{n_i}\sum_{k=0}^{n_i-1} f \circ T^k(y_i)$. Take functionals $\mu_i(g) = \liminf_i \frac{1}{m_i}\sum_{k=0}^{m_i-1} g \circ T^k(x)$ and $\nu_i(g) = \liminf_i \frac{1}{n_i}\sum_{k=0}^{n_i-1} g \circ T^k(x)$. Both sequences have weak accumulation points μ and ν which are easily shown to be T-invariant measures, see the proof of Theorem 1. But they are not the same because $\mu(f) = A \neq B = \nu(f)$. □

4 The Ergodic Theorem

Theorem 2 is an instance of a very general fact observed in ergodic theory:

Space Average = Time Average (for typical points).

This is expressed in the

Theorem 3 (Birkhoff Ergodic Theorem)**.** *Let μ be a probability measure and $\psi \in L^1(\mu)$. Then the* **ergodic average**

$$\overline{\psi}(x) := \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \psi \circ T^i(x)$$

exists μ-a.e. (everywhere if ψ is continuous), and $\overline{\psi}$ is T-invariant, i.e., $\overline{\psi} \circ T = \overline{\psi}$ *μ-a.e. If in addition μ is ergodic then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi \circ T^i(x) = \int_X \psi \, d\mu \qquad \mu\text{-a.e.} \tag{2}$$

Remark 2. *A point $x \in X$ satisfying* (2) *is called* **typical** *for μ. To be precise, the set of μ-typical points also depends on ψ, but for different functions $\psi, \tilde{\psi}$, the (μ, ψ)-typical points and $(\mu, \tilde{\psi})$-typical points differ only on a null-set.*

Corollary 1. *Lebesgue measure is the only R_α-invariant probability measure.*

Proof. Suppose R_α had two invariant measures, μ and ν. Then there must be an interval J such that $\mu(J) \neq \nu(J)$. Let $\psi = 1_J$ be the indicator function; it will belongs to $L^1(\mu)$ and $L^1(\nu)$. Apply Birkhoff's Ergodic Theorem for some μ-typical point x and ν-typical point y. Since their visit frequencies to J are the same, we have

$$\begin{aligned} \mu(J) &= \int \psi \, d\mu = \lim_n \frac{1}{n} \#\{0 \leqslant i < n : R_\alpha(x) \in J\} \\ &= \lim_n \frac{1}{n} \#\{0 \leqslant i < n : R_\alpha(y) \in J\} = \int \psi \, d\nu = \nu(J), \end{aligned}$$

a contradiction to μ and ν being different. ☐

5 Absolute continuity and invariant densities

Definition 4. *A measure μ is called* **absolutely continuous** *w.r.t. the measure ν (notation: $\mu \ll \nu$ if $\nu(A) = 0$ implies $\mu(A) = 0$. If both $\mu \ll \nu$ and $\nu \ll \mu$, then μ and ν are called equivalent.*

Proposition 3. *If $\mu \ll \nu$ are both T-invariant probability measures, with a common σ-algebra $\mathcal{B}$ of measurable sets. If ν is ergodic, then $\mu = \nu$.*

Proof. First we show that μ is ergodic. Indeed, otherwise there is a T-invariant set A such that $\mu(A) > 0$ and $\mu(A^c) > 0$. By ergodicity of ν at least one of A or A^c must have ν-measure 0, but this would contradict that $\mu \ll \nu$.

Now let $A \in \mathcal{B}$ and let $Y \subset X$ be the set of ν-typical points. Then $\nu(Y^c) = 0$ and hence $\mu(Y^c) = 0$. Applying Birkhoff's Ergodic Theorem to μ and ν separately for $\psi = 1_A$ and some μ-typical $y \in Y$, we get

$$\mu(A) = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \psi \circ T(y) = \nu(A).$$

But $A \in \mathcal{B}$ was arbitrary, so $\mu = \nu$. ☐

Theorem 4 (Radon-Nikodym)**.** *If μ is a probability measure and $\mu \ll \nu$ then there is a function $h \in L^1(\nu)$ (called* **Radon-Nikodym derivative** *or* **density***) such that $\mu(A) = \int_A h(x)\ d\nu(x)$ for every measurable set A.*

Sometimes we use the notation: $h = \frac{d\mu}{d\nu}$.

Proposition 4. *Let $T : U \subset \mathbb{R}^n \to U$ be (piecewise) differentiable, and μ is absolutely continuous w.r.t. Lebesgue. Then μ is T-invariant if and only if its density $h = \frac{d\mu}{dx}$ satisfies*

$$h(x) = \sum_{T(y)=x} \frac{h(y)}{|\det DT(y)|} \tag{3}$$

for every x.

Proof. The T-invariance means that $d\mu(x) = d\mu(T^{-1}(x))$, but we need to beware that T^{-1} is multivalued. So it is more careful to split the space U into pieces U_n such that the restrictions $T_n := T|U_n$ are diffeomorphism (onto their images) and write $y_n = T_n^{-1}(x) = T^{-1}(x) \cap U_n$. Then we obtain (using the change of coordinates)

$$\begin{aligned} h(x)\ dx &= d\mu(x) = d\mu(T^{-1}(x)) = \sum_n d\mu \circ T_n^{-1}(x) \\ &= \sum_n h(y_n) |\det(DT_n^{-1})(x)| dy_n = \sum_n \frac{h(y_n)}{\det |DT(y_n)|} dy_n, \end{aligned}$$

and the statement follows.

Conversely, if (3) holds, then the above computation gives $d\mu(x) = d\mu \circ T^{-1}(x)$, which is the required invariance. □

Example: If $T : [0,1] \to [0,1]$ is (countably) piecewise linear, and each branch $T : I_n \to [0,1]$ (on which T is affine) is onto, then T preserves Lebesgue measure. Indeed, the intervals I_n have pairwise disjoint interiors, and their lengths add up to 1. If s_n is the slope of $T : I_n \to [0,1]$, then $s_n = 1/|I_n|$. Therefore $\sum_n \frac{1}{DT(y_n)} = \sum_n 1/s_n = \sum_n |I_n| = 1$.

Example: The map $T : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ T(x) = x - \frac{1}{x}$ is called the **Boole transformation**. It is 2-to-1; the two preimages of $x \in \mathbb{R}$ are $y_\pm = \frac{1}{2}(x \pm \sqrt{x^2+4})$. Clearly $T'(x) = 1 + \frac{1}{x^2}$. A tedious computation shows that

$$\frac{1}{|T'(y_-)|} + \frac{1}{|T'(y_+)|} = 1.$$

Indeed, $|T'(y_\pm)| = 1 + \frac{2}{x^2+2\pm x\sqrt{x^2+4}}$, $1/|T'(y_\pm)| = \frac{x^2+2\pm x\sqrt{x^2+4}}{x^2+4\pm x\sqrt{x^2+4}}$, and

$$\begin{aligned}
\frac{1}{|T'(y_-)|} + \frac{1}{|T'(y_+)|} &= \frac{x^2+2-x\sqrt{x^2+4}}{x^2+4-x\sqrt{x^2+4}} + \frac{x^2+2+x\sqrt{x^2+4}}{x^2+4+x\sqrt{x^2+4}} \\
&= \frac{(x^2+2-x\sqrt{x^2+4})(x^2+4+x\sqrt{x^2+4})}{(x^2+4)^2 - x^2(x^2+4)} \\
&\qquad + \frac{(x^2+2+x\sqrt{x^2+4})(x^2+4-x\sqrt{x^2+4})}{(x^2+4)^2 - x^2(x^2+4)} \\
&= \frac{(x^2+2)^2 - x^2(x^2+4) + 2(x^2+2) - 2x\sqrt{x^2+4}}{4(x^2+4)} + \\
&\qquad \frac{(x^2+2)^2 - x^2(x^2+4) + 2(x^2+2) + 2x\sqrt{x^2+4}}{4(x^2+4)} \\
&= \frac{4(x^2+2)+8}{4(x^2+4)} = 1.
\end{aligned}$$

Therefore $h(x) \equiv 1$ is an invariant density, so Lebesgue measure is preserved.

Example: The **Gauß map** $G : (0,1] \to [0,1)$ is defined as $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$. It has an invariant density $h(x) = \frac{1}{\log 2}\frac{1}{1+x}$. Here $\frac{1}{\log 2}$ is just the normalising factor (so that $\int_0^1 h(x)dx = 1$).

Let $I_n = (\frac{1}{n+1}, \frac{1}{n}]$ for $n = 1, 2, 3, \dots$ be the domains of the branches of G, and for $x \in (0,1)$, and $y_n := G^{-1}(x) \cap I_n = \frac{1}{x+n}$. Also $G'(y_n) = -\frac{1}{y_n^2}$. Therefore

$$\begin{aligned}
\sum_{n\geqslant 1} \frac{h(y_n)}{|G'(y_n)|} &= \frac{1}{\log 2} \sum_{n\geqslant 1} \frac{y_n^2}{1+y_n} = \frac{1}{\log 2} \sum_{n\geqslant 1} \frac{\frac{1}{(x+n)^2}}{1+\frac{1}{x+n}} \\
&= \frac{1}{\log 2} \sum_{n\geqslant 1} \frac{1}{x+n}\frac{1}{x+n+1} \\
&= \frac{1}{\log 2} \sum_{n\geqslant 1} \frac{1}{x+n} - \frac{1}{x+n+1} \qquad \text{telescoping series} \\
&= \frac{1}{\log 2}\frac{1}{x+1} = h(x).
\end{aligned}$$

Exercise 1. *Show that for each integer $n \geqslant 2$, the interval map given by*

$$T_n(x) = \begin{cases} nx & \text{if } 0 \leqslant x \leqslant \frac{1}{n}, \\ \frac{1}{x} - \lfloor \frac{1}{x} \rfloor & \text{if } \frac{1}{n} < x \leqslant 1, \end{cases}$$

has invariant density $\frac{1}{\log 2}\frac{1}{1+x}$.

Theorem 5. *If $T : \mathbb{S}^1 \to \mathbb{S}^1$ is a C^2 expanding circle map, then it preserves a measure μ equivalent to Lebesgue, and μ is ergodic.*

Expanding here means that there is $\lambda > 1$ such that $|T'(x)| \geqslant \lambda$ for all $x \in \mathbb{S}^1$. The above theorem can be proved in more generality, but in the stated version it conveys the ideas more clearly.

Proof. First some estimates on derivatives. Using the Mean Value Theorem twice, we obtain

$$\begin{aligned}
\log \frac{|T'(x)|}{|T'(y)|} &= \log(1 + \frac{|T'(x)| - |T'(y)|}{|T'(y)|}) \leqslant \frac{|T'(x)| - |T'(y)|}{|T'(y)|} \\
&= \frac{|T''(\xi)| \cdot |x-y|}{|T'(y)|} = \frac{|T''(\xi)|}{|T'(y)|}\frac{|Tx - Ty|}{\lambda} \\
&\leqslant \sup_{\zeta} \frac{|T'(\zeta)|}{|T'(y)|}\frac{|Tx-Ty|}{\lambda} \leqslant K|Tx - Ty|
\end{aligned}$$

for some constant K. The chain rule then gives:

$$\log \frac{|DT^n(x)|}{|DT^n(y)|} = \sum_{i=0}^{n-1} \log \frac{|T'(T^i x)|}{|T'(T^i y)|} \leqslant K \sum_{i=1}^{n} |T^i(x) - T^i(y)|.$$

Since T is a continuous expanding map of the circle, it wraps the circle d times around itself, and for each n, there are d^n pairwise disjoint intervals Z such that $T^i Z \to \mathbb{S}^1$ is onto, with slope at least λ^i. If we take x, y above in one such Z, then $|x - y| < \lambda^{-n}|T^n(x) - T^n(y)|$ and in fact $|T^i(x) - T^i(y)| < \lambda^{-(n-i)}|T^n(x) - T^n(y)|$. Therefore we obtain

$$\log \frac{|DT^n(x)|}{|DT^n(y)|} = K \sum_{i=1}^{n} \lambda^{-(n-i)} |T^n(x) - T^n(y)| \leqslant \frac{K}{\lambda - 1}|T^n(x) - T^n(y)| \leqslant \log K'$$

for some $K' \in (1, \infty)$. This means that if $A \subset Z$ (so $T^n : A \to T^n(A)$ is a bijection), then

$$\frac{1}{K'}\frac{m(A)}{m(Z)} \leqslant \frac{m(T^n A)}{m(T^n Z)} = \frac{m(T^n A)}{m(\mathbb{S}^1)} \leqslant K'\frac{m(A)}{m(Z)}, \tag{4}$$

where m is Lebesgue measure.

Now we construct the T-invariant measure μ. Take $B \subset \mathcal{B}$ arbitrary, and set $\mu_n(B) = \frac{1}{n}\sum_{i=0}^{n-1} m(T^{-i}B)$. Then by (4),

$$\frac{1}{K'} m(B) \leqslant \mu_n(B) \leqslant K' m(B).$$

We can take a weak* limit of the μ_n's, call it μ, then

$$\frac{1}{K'} m(B) \leqslant \mu(B) \leqslant K' m(B),$$

and therefore μ and m are equivalent. The T-invariance of μ proven in the same way as Theorem 1.

Now for the ergodicity of μ, we need the Lebesgue Density Theorem, which says that if $m(A) > 0$, then for m-a.e. $x \in A$, the limit

$$\lim_{\varepsilon \to 0} \frac{m(A \cap B_\varepsilon(x))}{m(B_\varepsilon(x))} = 1,$$

where $B_\varepsilon(x)$ is the ε-balls around x. Points x with this property are called **(Lebesgue) density points** of A. (In fact, the above also holds, if $B_\varepsilon(x)$ is just a one-sided ε-neighbourhood of x.)

Assume by contradiction that μ is not ergodic. Take $A \in \mathcal{B}$ a T-invariant set such that $\mu(A) > 0$ and $\mu(A^c) > 0$. By equivalence of μ and m, also $\delta := m(A^c) > 0$. Let x be a density point of A, and Z_n be a neighbourhood of x such that $T^n : Z \to \mathbb{S}^1$ is a bijection. As $n \to \infty$, $Z \to \{x\}$, and therefore we can choose n so large (hence Z so small) that

$$\frac{m(A \cap Z)}{m(Z)} > 1 - \delta/K'.$$

Therefore $\frac{m(A^c \cap Z)}{m(Z)} < \delta/K'$, and using (4),

$$\frac{m(T^n(A_c \cap Z))}{m(T^n(Z))} \leqslant K' \frac{m(A^c \cap Z)}{m(Z)} < K'\delta/K' = \delta.$$

Since $T^n : A^c \cap Z \to A^c$ is a bijection, and $m(T^n Z) = m(\mathbb{S}^1) = 1$, we get $\delta = m(A^c) < \delta$, a contraction. Therefore μ is ergodic. □

6 The Choquet Simplex and the Ergodic Decomposition

Throughout this section, let $T : X \to X$ a **continuous** transformation of a compact metric space. Recall that $\mathcal{M}(X)$ is the collection of probability measures defined on X; we saw in (1) that it is compact in the weak* topology. In general, X carries many T-invariant measures. The set $\mathcal{M}(X,T) = \{\mu \in \mathcal{M}(X) : \mu \text{ is } T\text{-invariant}\}$ is called the **Choquet simplex** of T. Let $\mathcal{M}_{erg}(X,T)$ be the subset of $\mathcal{M}(X,T)$ of ergodic T-invariant measures.

Clearly $\mathcal{M}(X,T) = \{\mu\}$ if (X,T) is uniquely ergodic. The name "simplex" just reflects the convexity of $\mathcal{M}(X,T)$: if $\mu_1, \mu_2 \in \mathcal{M}(X,T)$, then also $\alpha\mu_1 + (1-\alpha)\mu_2 \in \mathcal{M}(X,T)$ for every $\alpha \in [0,1]$.

Lemma 2. *The Choquet simplex $\mathcal{M}(X,T)$ is a compact subset of $\mathcal{M}(X)$ w.r.t. weak* topology.*

Proof. Suppose $\{\mu_n\} \subset \mathcal{M}(X,T)$, then by the compactness of $\mathcal{M}(X)$, see (1), there is $\mu \in \mathcal{M}(X)$ and a subsequence $(n_i)_i$ such that for every continuous function $f : X \to \mathbb{R}$ such that $\int f d\mu_{n_i} \to \int f d\mu$ as $i \to \infty$. It remains to show that μ is T-invariant, but this simply follows from continuity of $f \circ T$ and

$$\int f \circ T \, d\mu = \lim_i \int f \circ T \, d\mu_{n_i} = \lim_i \int f \, d\mu_{n_i} = \int f \, d\mu.$$

□

Theorem 6. *The ergodic measures are exactly the extremal points of the Choquet simplex.*

Proof. First assume that μ is not ergodic. Hence there is a T-invariant set A such that $0 < \mu(A) < 1$. Define

$$\mu_1(B) = \frac{\mu(B \cap A)}{\mu(A)} \quad \text{and} \quad \mu_2(B) = \frac{\mu(B \setminus A)}{\mu(X \setminus A)}.$$

Then $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for $\alpha = \mu(A) \in (0,1)$ so μ is not an extremal point.

Suppose now that μ is ergodic but that $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for some $\alpha \in (0,1)$. We need to show that $\mu_1 = \mu_2 = \mu$. From the definition, it is clear that $\mu_1 \ll \mu$, so a Radon-Nikodym derivative $\frac{d\mu_1}{d\mu}$ exists in $L^1(\mu)$. Let $A^- = \{x \in X : \frac{d\mu_1}{d\mu} < 1\}$. Then

$$\begin{aligned} \int_{A^- \cap T^{-1}A^-} \frac{d\mu_1}{d\mu} \, d\mu + \int_{A^- \setminus T^{-1}A^-} \frac{d\mu_1}{d\mu} \, d\mu = \mu_1(A^-) \\ = \mu_1(T^{-1}A^-) = \int_{T^{-1}A^- \cap A^-} \frac{d\mu_1}{d\mu} \, d\mu + \int_{T^{-1}A^- \setminus A^-} \frac{d\mu_1}{d\mu} \, d\mu. \end{aligned}$$

Canceling the term $\int_{A^- \cap T^{-1}A^-} \frac{d\mu_1}{d\mu} \, d\mu$ gives

$$\int_{A^- \setminus T^{-1}A^-} \frac{d\mu_1}{d\mu} \, d\mu = \int_{T^{-1}A^- \setminus A^-} \frac{d\mu_1}{d\mu} \, d\mu. \tag{5}$$

But also $\mu(T^{-1}A^- \setminus A^-) = \mu(T^{-1}A^-) - \mu(T^{-1}A^- \cap A^-) = \mu(A^- \setminus T^{-1}A^-)$. Therefore, in (5), both integrations are over sets of the same measure, but in the left-hand side, the integrand < 1 while in the right-hand side, the integrand $\geqslant 1$. Therefore $\mu(T^{-1}A^- \setminus A^-) = \mu(A^- \setminus T^{-1}A^-) = 0$, and hence A^- is T-invariant. By assumed ergodicity of μ, $\mu(A^-) = 0$ or 1. In the latter case,

$$1 = \mu_1(X) = \int_X \frac{d\mu_1}{d\mu} \, d\mu = \int_{A^-} \frac{d\mu_1}{d\mu} \, d\mu < \mu(A^-) = 1,$$

a contradiction. Therefore $\mu(A^-) = 0$. But then we can repeat the argument for $A^+ = \{x \in X : \frac{d\mu_1}{d\mu} > 1\}$ and find that $\mu(A^+) = 0$ as well. Therefore $\frac{d\mu_1}{d\mu} = 1$ μ-a.e. and hence $\mu_1 = \mu$. But then also $\mu_2 = \mu$, which finishes the proof. □

The following fundamental theorem implies that for checking the properties of any measure $\mu \in \mathcal{M}(X,T)$, it suffices to verify the properties for ergodic measures:

Theorem 7 (Ergodic Decomposition). *For every $\mu \in \mathcal{M}(X,T)$, there is a measure ν on the spaces of ergodic measures such that $\nu(\mathcal{M}_{erg}(X,T)) = 1$ and*

$$\mu(B) = \int_{\mathcal{M}_{erg}(X,T)} m(B)\ d\nu(m)$$

for all Borel sets B.

7 Poincaré Recurrence

Theorem 8 (Poincaré's Recurrence Theorem). *If (X,T,μ) is a measure preserving system with $\mu(X)=1$, then for every measurable set $U \subset X$ of positive measure, μ-a.e. $x \in U$ returns to U,* i.e., *there is $n = n(x)$ such that $T^n(x) \in U$.*

Proof of Theorem 8. Let U be an arbitrary measurable set of positive measure. As μ is invariant, $\mu(T^{-i}(U)) = \mu(U) > 0$ for all $i \geqslant 0$. On the other hand, $1 = \mu(X) \geqslant \mu(\cup_i T^{-i}(U))$, so there must be overlap in the backward iterates of U, *i.e.,* there are $0 \leqslant i < j$ such that $\mu(T^{-i}(U) \cap T^{-j}(U)) > 0$. Take the j-th iterate and find $\mu(T^{j-i}(U) \cap U) \geqslant \mu(T^{-i}(U) \cap T^{-j}(U)) > 0$. This means that a positive measure part of the set U returns to itself after $n := j-i$ iterates.

For the part U' of U that didn't return after n steps, assuming U' has positive measure, we repeat the argument. That is, there is n' such that $\mu(T^{n'}(U') \cap U') > 0$ and then also $\mu(T^{n'}(U') \cap U) > 0$.

Repeating this argument, we can exhaust the set U up to a set of measure zero, and this proves the theorem. □

Definition 5. *A system $(X,T,\mathcal{B},\mu)$ is called* conservative *if for every set $A \in \mathcal{B}$ with $\mu(A) > 0$, there is $n \geqslant 1$ such that $\mu(T^n(A) \cap A) > 0$. The system is called* dissipative *otherwise, and it is called* totally dissipative *if $\mu(T^n(A) \cap A) = 0$ for very set $A \in \mathcal{B}$.*

We call the transformation T **recurrent** *w.r.t. μ if $B \setminus \cup_{i\in\mathbb{N}} T^{-i}(B)$ has zero measure for every $B \in \mathcal{B}$. In fact, this is equivalent to μ being conservative.*

The Poincaré Recurrence Theorem thus states that probability measure preserving systems are conservative.

Lemma 3 (Kac Lemma). *Let (X,T) preserve an ergodic measure μ. Take $Y \subset X$ measurable such that $0 < \mu(Y) \leqslant 1$, and let $\tau : Y \to \mathbb{N}$ be the first return time to Y.*

Then

$$\int \tau d\mu = \sum_{k\geqslant 1} k\mu(Y_k) = \begin{cases} 1 & \textit{if } \mu \textit{ is a probability measure },\\ \infty & \textit{if } \mu \textit{ is a conservative } \sigma\textit{-finite measure.}\end{cases}$$

for $Y_k := \{y \in Y : \tau(y) = k\}$.

Proof. Build a tower over Y by defining levels $L_0 = Y$, $L_1 = T(Y) \setminus Y$ and recursively $L_{j+1} = T(L_j) \setminus Y$. Then $L_j = \{T^j(y) : y \in Y, T^k(x) \notin Y \text{ for } 0 < k < j\}$. In particular, all the L_j are disjoint and $T(L_j) \subset L_{j+1} \cup Y$.

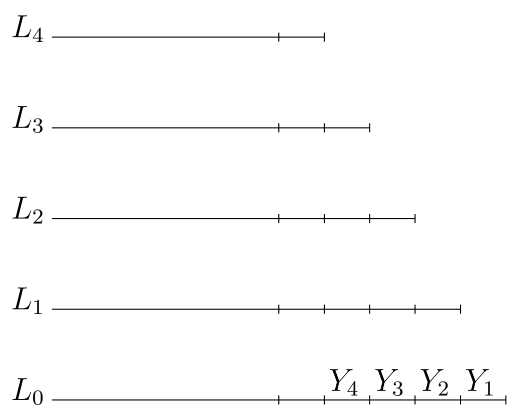

Figure 1: The tower consisting of levels L_j, $j \geqslant 0$.

We claim that $B := \cup_{j\geqslant 0} L_j$ is T-invariant (up to measure zero). Clearly $T^{-1}(L_j) \subset L_{j-1}$ for $j \geqslant 1$. Hence, we only need to show that $T^{-1}(Y) \subset B \pmod{\mu}$. Set $A := T^{-1}(Y) \setminus B$. We consider the two cases:

- $\mu(X) = 1$: if $x \in A$, then $T^{-j}(x) \notin Y$ for all $j \geqslant 0$, because if $j \geqslant 0$ were the minimal value such that $T^j(z) = x$ for some $z \in Y$, then $x \in L_j$.

  The sets $T^{-j}(A)$, $j \geqslant 0$, are in fact pairwise disjoint because if $x \in T^{-j}(A) \cap T^{-k}(A)$ for some minimal $0 \leqslant j < k$, then $T^{j-k}(A) \subset L_{2k-j-1}$, contradicting the previous paragraph.

  But this means that if $\mu(A) > 0$, then not only $\mu(T^{-j}(A)) = \mu(A) > 0$, but by disjointness, $\mu(\cup_j T^{-j}(A) = \sum_j \mu(T^{-j}(A)) = \infty$, contradicting that μ is a probability measure.

  This proves that $\mu(A) = 0$, so $T^{-1}(B) = B \pmod{\mu}$ and by ergodicity, $\mu(B) = 1$.

- $\mu(X) = \infty$ and μ is conservative: Note that $T^k(A) \cap A = \varnothing$ for all $k \geqslant 1$. Therefore, if $\mu(A) > 0$, we have a contradiction to conservativity.

The sets Y_k are clearly pairwise disjoint. Since $\tau(y) < \infty$ for μ-a.e. $y \in Y$, $\cup_k Y_k = Y \pmod{\mu}$. Furthermore, $T^j(Y_k)$ are pairwise disjoint subsets of L_j for $j < k$ and $L_j = \cup_{k>j} T^j(Y_k) \pmod{\mu}$. Finally, $T^{-1}(T^j(Y_k) \cap L_j) = T^{j-1}(Y_k) \cap L_{j-1}$ for $1 \leqslant j < k$. By T-invariance,

$$\mu(T^j(Y_k) \cap L_j) = \mu(T^{j-1}(Y_k) \cap L_{j-1}) = \cdots = \mu(T(Y_k) \cap L_1) = \mu(Y_k)$$

for $0 \leqslant j < k$.

Therefore (swapping the order of summation in the second line)

$$\begin{aligned} \mu(X) = \mu(B) &= \sum_{j \geqslant 0} \mu(L_j) = \sum_{j \geqslant 0} \sum_{k > j} \mu(T^j(Y_k) \cap L_j) \\ &= \sum_{k \geqslant 1} \sum_{0 \leqslant j < k} \mu(T^j(Y_k) \cap L_j) \\ &= \sum_{k \geqslant 1} \sum_{0 \leqslant j < k} \mu(Y_k) = \sum_{k \geqslant 1} k \mu(Y_k), \end{aligned}$$

as required. □

8 The Koopman operator

Given a probability measure preserving dynamical system $(X, \mathcal{B}, \mu, T)$, we can take the space of complex-valued square-integrable observables $L^2(\mu)$. This is a Hilbert space, equipped with inner product $\langle f, g \rangle = \int_X f(x) \cdot \overline{g(x)} \, d\mu$.

The Koopman operator $U_T : L^2(\mu) \to L^2(\mu)$ is defined as $U_T f = f \circ T$. By T-invariance of μ, it is a unitary operator. Indeed

$$\langle U_T f, U_T g \rangle = \int_X f \circ T(x) \cdot \overline{g \circ T(x)} \, d\mu = \int_X (f \cdot \overline{g}) \circ T(x) \, d\mu = \int_X f \cdot \overline{g} \, d\mu = \langle f, g \rangle,$$

and therefore $U_T^* U_T = U_T U_T^* = I$. This has several consequences, common to all unitary operators. First of all, the spectrum $\sigma(U_T)$ of U_T is a closed subset of the unit circle.

Secondly, we can give a (continuous) decomposition of U_T in orthogonal projections, called spectral decomposition. For a fixed eigenfunction ψ (with eigenvalue $\lambda \in \mathbb{S}^1$, we let $\Pi_\lambda : L^2(\mu) \to L^2(\mu)$ be the orthogonal projection onto the span of ψ. More generally, if $S \subset \sigma(U_T)$, we define Π_S as the orthogonal projection on the largest closed subspace V such that $U_T|_V$ has spectrum contained in S. As any orthogonal projection, we have the properties:

- $\Pi_S^2 = \Pi_S$ (Π_S is idempotent);

- $\Pi_S^* = \Pi_S$ (Π_S is self-adjoint);
- $\Pi_S \Pi_{S'} = 0$ if $S \cap S' = \varnothing$;
- The kernel $\mathcal{N}(\Pi_S)$ equals the orthogonal complement, $V^\perp$, of V.

Theorem 9 (Spectral Decomposition of Unitary Operators)**.** *There is a measure ν_T on $\mathbb{S}^1$ such that*

$$U_T = \int_{\sigma(U_T)} \lambda \, \Pi_\lambda d\nu_T(\lambda),$$

and $\nu_T(\lambda) \neq 0$ if and only if λ is an eigenvalue of U_T. Using the above properties of orthogonal projections, we also get

$$U_T^n = \int_{\sigma(U_T)} \lambda^n \Pi_\lambda d\nu_T(\lambda).$$

9 Bernoulli shifts

Let (Σ, σ, μ) be a Bernoulli shift, say with alphabet $\mathcal{A} = \{1, 2, \dots, N\}$. Here $\Sigma = \mathcal{A}^{\mathbb{Z}}$ (two-sided) or $\Sigma = \mathcal{A}^{\mathbb{N} \cup \{0\}}$ (one-sided), and μ is a stationary product measure with probability vector $(p_1, \dots, p_N)$. Write

$$Z_{[k+1,k+N]}(a_1 \dots a_N) = \{x \in \Sigma : x_{k+1} \dots x_{k+N} = a_1 \dots a_N\}$$

for the cylinder set of length N. If $C = Z_{[k+1,k+R]}$ and $C' = Z_{[l+1,l+S]}$ are two cylinders fixing coordinates on disjoint integer intervals (*i.e.,* $[k+1, k+R] \cap [l+1, l+S] = \varnothing$), then clearly $\mu(C \cap C') = \mu(C)\mu(C')$. This just reflects the independence of disjoint events in a sequence of Bernoulli trials.

Definition 6. *Two measure preserving dynamical systems $(X, \mathcal{B}, T, \mu)$ and $(Y, \mathcal{C}, S, \nu)$ are called* **isomorphic** *if there are $X' \in \mathcal{B}$, $Y' \in \mathcal{C}$ and $\phi : Y' \to X'$ such that*

- $\mu(X') = 1$, $\nu(Y') = 1$;
- $\phi : Y' \to X'$ *is a bi-measurable bijection;*
- ϕ *is measure preserving:* $\nu(\phi^{-1}(B)) = \mu(B)$ *for all* $B \in \mathcal{B}$.
- $\phi \circ S = T \circ \phi$.

Clearly invertible systems cannot be isomorphic to non-invertible systems. But there is a construction to make a non-invertible system invertible, namely by passing to the natural extension.

Definition 7. *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving dynamical system. A system $(Y, \mathcal{C}, S, \nu)$ is a* **natural extension** *of $(X, \mathcal{B}, \mu, T)$ if there are $X' \in \mathcal{B}$, $Y' \in \mathcal{C}$ and $\phi : Y' \to X'$ such that*

- $\mu(X') = 1$, $\nu(Y') = 1$;
- $S : Y' \to Y'$ *is* **invertible**;
- $\phi : Y' \to X'$ *is a measurable surjection;*
- ϕ *is measure preserving:* $\nu(\phi^{-1}(B)) = \mu(B)$ *for all* $B \in \mathcal{B}$;
- $\phi \circ S = T \circ \phi$.

Any two natural extensions can be shown to be isomorphic, so it makes sense to speak of **the** natural extension. Sometimes natural extensions have explicit formulas (such as the baker transformation being the natural extension of the angle doubling map). There is also a general construction: Set

$$Y = \{(x_i)_{i \geqslant 0} : T(x_{i+1}) = x_i \in X \text{ for all } i \geqslant 0\}$$

with $S(x_0, x_1, \dots) = T(x_0), x_0, x_1, \dots$. Then S is invertible (with the left shift $\sigma = S^{-1}$) and

$$\nu(A_0, A_1, A_2, \dots) = \inf_i \mu(A_i) \quad \text{for } (A_0, A_1, A_2 \dots) \subset S,$$

is S-invariant. Now defining $\phi(x_0, x_1, x_2, \dots) := x_0$ makes the diagram commute: $T \circ \phi = \phi \circ S$. Also ϕ is measure preserving because, for each $A \in \mathcal{B}$,

$$\phi^{-1}(A) = (A, T^{-1}(A), T^{-2}(A), T^{-3}(A), \dots)$$

and clearly $\nu(A, T^{-1}(A), T^{-2}(A), T^{-3}(A), \dots) = \mu(A)$ because $\mu(T^{-i}(A)) = \mu(A)$ for every i by T-invariance of μ.

Definition 8. *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving dynamical system.*

- *If T is invertible, then the system is called* **Bernoulli** *if it is isomorphic to a Bernoulli shift.*
- *If T is non-invertible, then the system is called* **one-sided Bernoulli** *if it is isomorphic to a one-sided Bernoulli shift.*
- *If T is non-invertible, then the system is called* **Bernoulli** *if its natural extension is isomorphic to a one-sided Bernoulli shift.*

The Bernoulli property is quite general, even though the isomorphism ϕ may be very difficult to find explicitly. Expanding circle maps that satisfy the conditions of Theorem 5 are also Bernoulli, *i.e.,* have a Bernoulli natural extension, see [11]. Being **one-sided Bernoulli**, on the other hand quite, is special. If $T : [0,1] \to [0,1]$ has N linear surjective branches I_i, $i = 1, \dots, N$, then Lebesgue measure m is invariant, and $([0,1], \mathcal{B}, m, T)$ is isomorphic to the one-sided Bernoulli system with probability vector $(|I_1|, \dots, |I_N|)$. If T is piecewise C^2 but not piecewise linear, then it has to be C^2-conjugate to a piecewise linear expanding map to be one-sided Bernoulli, see [6].

10 Mixing and weak mixing

Whereas Bernoulli trials are totally independent, mixing refers to an **asymptotic independence**:

Definition 9. *A probability measure preserving dynamical systems* $(X, \mathcal{B}, \mu, T)$ *is* **mixing** *(or* **strong mixing***) if*

$$\mu(T^{-n}(A) \cap B) \to \mu(A)\mu(B) \text{ as } n \to \infty \tag{6}$$

for every $A, B \in \mathcal{B}$*.*

Proposition 5. *A probability preserving dynamical systems* $(X, \mathcal{B}, T, \mu)$ *is mixing if and only if*

$$\int_X f \circ T^n(x) \cdot \overline{g(x)}\ d\mu \to \int_X f(x)\ d\mu \cdot \int_X \overline{g(x)}\ d\mu \text{ as } n \to \infty \tag{7}$$

for all $f, g \in L^2(\mu)$*, or written in the notation of the Koopman operator* $U_T f = f \circ T$ *and inner product* $\langle f, g \rangle = \int_X f(x) \cdot \overline{g(x)}\ d\mu$*:*

$$\langle U_T^n f, g \rangle \to \langle f, 1 \rangle \langle 1, g \rangle \text{ as } n \to \infty. \tag{8}$$

Proof. The "if"-direction follows by taking indicator functions $f = 1_A$ and $g = 1_B$. For the "only if"-direction, general $f, g \in L^2(\mu)$ can be approximated by linear combinations of indicator functions. □

Definition 10. *A probability measure preserving dynamical systems* $(X, \mathcal{B}, \mu, T)$ *is* **weak mixing** *if in average*

$$\frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)| \to 0 \text{ as } n \to \infty \tag{9}$$

for every $A, B \in \mathcal{B}$*.*

We can express ergodicity in analogy of (6) and (9):

Lemma 4. *A probability preserving dynamical systems $(X, \mathcal{B}, T, \mu)$ is ergodic if and only if*

$$\frac{1}{n}\sum_{i=0}^{n-1} \mu(T^{-i}(A) \cap B) - \mu(A)\mu(B) \to 0 \text{ as } n \to \infty,$$

for all $A, B \in \mathcal{B}$. (Compared to (9), note the absence of absolute value bars.)

Proof. Assume that T is ergodic, so by Birkhoff's Ergodic Theorem $\frac{1}{n}\sum_{i=0}^{n-1} 1_A \circ T^i(x) \to \mu(A)$ for μ-a.e. x. Multiplying by 1_B gives

$$\frac{1}{n}\sum_{i=0}^{n-1} 1_A \circ T^i(x) 1_B(x) \to \mu(A) 1_B(x) \quad \mu\text{-a.e.}$$

Integrating over x (using the Dominated Convergence Theorem to swap limit and integral), gives $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \int_X 1_A \circ T^i(x) 1_B(x)\, d\mu = \mu(A)\mu(B)$.

Conversely, assume that $A = T^{-1}A$ and take $B = A$. Then we obtain $\mu(A) = \frac{1}{n}\sum_{i=0}^{n-1} \mu(T^{-i}(A)) \to \mu(A)^2$, hence $\mu(A) \in \{0, 1\}$. □

Theorem 10. *We have the implications:*

Bernoulli ⇒ mixing ⇒ weak mixing ⇒ ergodic ⇒ recurrent.

None of the reverse implications holds in generality.

Proof. Bernoulli ⇒ mixing holds for any pair of cylinder sets C, C' because $\mu(\sigma^{-n}(C) \cap C) = \mu(C)\mu(C')$ for n sufficiently large. The property carries over to all measurable sets by the Kolmogorov Extension Theorem.

Mixing ⇒ weak mixing is immediate from the definition.

Weak mixing ⇒ ergodic: Let $A = T^{-1}(A)$ be a measurable T-invariant set. Then by weak mixing $\mu(A) = \frac{1}{n}\sum_{i=0}^{n-1} \mu(T^{-i}(A) \cap A) \to \mu(A)\mu(A) = \mu(A^2)$. This means that $\mu(A) = 0$ or 1.

Ergodic ⇒ recurrent. If $B \in \mathcal{B}$ has positive measure, then $A := \cup_{i \in \mathbb{N}} T^{-i}(B)$ is T-invariant up to a set of measure 0, see the Poincaré Recurrence Theorem. By ergodicity, $\mu(A) = 1$, and this is the definition of recurrence, see Definition 5. □

We say that a subset $E \subset \mathbb{N} \cup \{0\}$ has **density** zero if $\lim_n \frac{1}{n}\#(E \cap \{0, \dots, n-1\}) = 0$.

Lemma 5. *Let $(a_i)_{i \geqslant 0}$ be a bounded non-negative sequence of real numbers. Then $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} a_i = 0$ if and only if there is a sequence E of zero density in $\mathbb{N} \cup \{0\}$ such that $\lim_{E \not\ni n \to \infty} a_n = 0$.*

Proof. $\Leftarrow$: Assume that $\lim_{E\not\ni n\to\infty} a_n = 0$ and for $\varepsilon > 0$, take N such that $a_n < \varepsilon$ for all $E \not\ni n \geqslant N$. Also let $A = \sup a_n$. Then

$$\begin{aligned} 0 &\leqslant \frac{1}{n}\sum_{i=0}^{n-1} a_i = \frac{1}{n}\sum_{E\not\ni i=0}^{n-1} a_i + \frac{1}{n}\sum_{E\ni i=0}^{n-1} a_i \\ &\leqslant \frac{NA + (n-N)\varepsilon}{n} + A\frac{1}{n}\#(E \cap \{0,\dots,n-1\}) \to \varepsilon, \end{aligned}$$

as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary, $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} a_i = 0$.

$\Rightarrow$: Let $E_m = \{n : a_n \geqslant \frac{1}{m}\}$. Then clearly $E_1 \subset E_2 \subset E_3 \subset \dots$ and each E_m has density 0 because

$$0 = m \cdot \lim_n \frac{1}{n}\sum_{i=0}^{n-1} a_i \geqslant \lim_n \frac{1}{n}\sum_{i=0}^{n-1} 1_{E_m}(i) = \lim_n \frac{1}{n}\#(E_m \cap \{0,\dots n-1\}).$$

Now take $0 = N_0 < N_1 < N_2 < \dots$ such that $\frac{1}{n}\#(E_m \cap \{0,\dots,n-1\}) < \frac{1}{m}$ for every $n \geqslant N_{m-1}$. Let $E = \cup_m (E_m \cap \{N_{m-1},\dots,N_m - 1\})$.

Then, taking $m = m(n)$ maximal such that $N_{m-1} < n$,

$$\begin{aligned} &\frac{1}{n}\ \#\ (E \cap \{0,\dots,n-1\}) \\ &\quad\leqslant \frac{1}{n}\#(E_{m-1} \cap \{0,\dots,N_{m-1}-1\}) + \frac{1}{n}\#(E_m \cap \{N_{m-1},\dots,n-1\}) \\ &\quad\leqslant \frac{1}{N_{m-1}}\#(E_{m-1} \cap \{0,\dots,N_{m-1}-1\}) + \frac{1}{n}\#(E_m \cap \{0,\dots,n-1\}) \\ &\quad\leqslant \frac{1}{m-1} + \frac{1}{m} \to 0 \end{aligned}$$

as $n \to \infty$. □

Corollary 2. *For a non-negative sequence $(a_n)_{n\geqslant 0}$ of real numbers, $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} a_i = 0$ if and only if $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} a_i^2 = 0$.*

Proof. By the previous lemma, $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} a_i = 0$ if and only if $\lim_{E\not\ni n\to\infty} a_n = 0$ for a set E of zero density. But the latter is clearly equivalent to $\lim_{E\not\ni n\to\infty} a_n^2 = 0$ for the same set E. Applying the lemma again, we have $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} a_i^2 = 0$. □

Example 1. *Let $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ be an irrational circle rotation; it preserves Lebesgue measure. We claim that R_α is not mixing or weak mixing, but it is ergodic. To see why R_α is not mixing, take an interval A of length $\frac{1}{4}$. There are infinitely many n such that $R_\alpha^{-n}(A) \cap A = \varnothing$, so $\liminf_n \mu(R^{-n}(A) \cap A) = 0 \neq (\frac{1}{4})^2$. However, R_α has a non-constant eigenfunction $\psi : \mathbb{S}^1 \to \mathbb{C}$ defined as $\psi(x) = e^{2\pi i x}$ because $\psi \circ R_\alpha(x) = e^{2\pi i(x+\alpha)} = e^{2\pi i\alpha}\psi(x)$. Therefore R_α is not weak mixing, see Theorem 11 below. To* *prove ergodicity, we show that every T-invariant function $\psi \in L^2(m)$ must be constant. Indeed, write $\psi(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$ as a Fourier series. The T-invariance implies that $a_n e^{2\pi i \alpha} = a_n$ for all $n \in \mathbb{Z}$. Since $\alpha \notin \mathbb{Q}$, this means that $a_n = 0$ for all $n \neq 0$, so $\psi(x) \equiv a_0$ is indeed constant.*

Theorem 11. *Let $(X, \mathcal{B}, \mu, T)$ be a probability measure preserving dynamical system. Then the following are equivalent:*

1. *$(X, \mathcal{B}, \mu, T)$ is weak mixing;*
2. *$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} |\langle f \circ T^i, g\rangle - \langle f, 1\rangle\langle 1, g\rangle| = 0$ for all $L^2(\mu)$ functions f, g;*
3. *$\lim_{E \not\ni n \to \infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B)$ for all $A, B \in \mathcal{B}$ and a subset E of zero density;*
4. *$T \times T$ is weak mixing;*
5. *$T \times S$ is ergodic on (X, Y) for every ergodic system $(Y, \mathcal{C}, \nu, S)$;*
6. *$T \times T$ is ergodic;*
7. *The Koopman operator U_T has no measurable eigenfunctions other than constants.*

Proof. 2. $\Rightarrow$ 1. Take $f = 1_A$, $g = 1_B$.
1. $\Leftrightarrow$ 3. Use Lemma 5 for $a_i = |\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)|$.
3. $\Rightarrow$ 4. For every $A, B, C, D \in \mathcal{B}$, there are subsets E_1 and E_2 of $\mathbb{N}$ of zero density such that

$$\lim_{E_1 \not\ni n \to \infty} \mu(T^{-n}(A) \cap B) - \mu(A)\mu(B)| = \lim_{E_2 \not\ni n \to \infty} \mu(T^{-n}(C) \cap D) - \mu(C)\mu(D)| = 0.$$

The union $E = E_1 \cup E_2$ still has density 0, and

$$\begin{aligned} 0 \leqslant \lim_{E \not\ni n \to \infty} | \ \mu \times \mu \ ((T \times T)^{-n}(A \times C) \cap (B \times D)) - \mu \times \mu(A \times B) \cdot \mu \times \mu(C \times D)| \\ = \quad \lim_{E \not\ni n \to \infty} |\mu(T^{-n}(A) \cap B) \cdot \mu(T^{-n}(C) \cap D) - \mu(A)\mu(B)\mu(C)\mu(D)| \\ \leqslant \quad \lim_{E \not\ni n \to \infty} \mu(T^{-n}(A) \cap B) \cdot |\mu(T^{-n}(C) \cap D) - \mu(C)\mu(D)| \\ + \lim_{E \not\ni n \to \infty} \mu(C)\mu(D) \cdot |\mu(T^{-n}(A) \cap B) - \mu(A)\mu(B)| = 0. \end{aligned}$$

4. $\Rightarrow$ 5. If $T \times T$ is weakly mixing, then so is T itself. Suppose $(Y, \mathcal{C}, \nu, S)$ is an ergodic

system, then, for $A, B \in \mathcal{B}$ and $C, D \in \mathcal{C}$ we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} \mu\left(T^{-i}(A) \cap B\right)\nu(S^{-i}(C) \cap D) \\
&= \frac{1}{n}\sum_{i=0}^{n-1} \mu(A)\mu(B)\nu(S^{-i}(C) \cap D) \\
&\quad + \frac{1}{n}\sum_{i=0}^{n-1} (\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B))\nu(S^{-i}(C) \cap D).
\end{aligned}$$

By ergodicity of S (see Lemma 4), $\frac{1}{n}\sum_{i=0}^{n-1} \nu(S^{-i}(C)\cap D) \to \mu(C)\mu(D)$, so the first term in the above expression tends to $\mu(A)\mu(B)\mu(C)\mu(D)$. The second term is majorised by $\frac{1}{n}\sum_{i=0}^{n-1} |\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)|$, which tends to 0 because T is weak mixing.

5. $\Rightarrow$ 6. By assumption $T \times S$ is ergodic for the trivial map $S : \{0\} \to \{0\}$. Therefore T itself is ergodic, and hence $T \times T$ is ergodic.

6. $\Rightarrow$ 7. Suppose f is an eigenfunction with eigenvalue λ. The Koopman operator is an isometry (by T-invariance of the measure), so $\langle f, f\rangle = \langle U_T f, U_T f\rangle = \langle \lambda f, \lambda f\rangle = |\lambda|^2\langle f, f\rangle$, and $|\lambda| = 1$. Write $\psi(x, y) = f(x)\overline{f}(y)$. Then

$$\psi \circ (T \times T)(x, y) = \psi(Tx, Ty) = f(Tx)\overline{f(Ty)} = |\lambda|^2\psi(x, y) = \psi(x, y),$$

so ψ is $T \times T$-invariant. By ergodicity of $T \times T$, ψ must be constant $\mu \times \mu$-a.e. But then also f must be constant μ-a.e.

7. $\Rightarrow$ 2. This is the hardest step; it relies on spectral theory of unitary operators. If ψ is an eigenfunction of U_T, then by assumption, ψ is constant, so the eigenvalue is 1. Let $V = \text{span}\ (\psi)$ and Π_1 is the orthogonal projection onto V; clearly $V^\perp = \{f \in L^2(\mu) : \int f\ d\mu = 0\}$. One can derive that the spectral measure ν_T cannot have any atoms, except possibly at Π_1.

Now take $f \in V^\perp$ and $g \in L^2(\mu)$ arbitrary. Using the Spectral Theorem 9, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} |\langle U_T^i f, g\rangle|^2 &= \frac{1}{n}\sum_{i=0}^{n-1} \left| \int_{\sigma(U_T)} \lambda^i \langle \Pi_\lambda f, g\rangle\ d\nu_T(\lambda) \right|^2 \\
&= \frac{1}{n}\sum_{i=0}^{n-1} \int_{\sigma(U_T)} \lambda^i \langle \Pi_\lambda f, g\rangle\ d\nu_T(\lambda) \overline{\int_{\sigma(U_T)} \kappa^i \langle \Pi_\kappa f, g\rangle\ d\nu_T(\kappa)} \\
&= \frac{1}{n}\sum_{i=0}^{n-1} \int\int_{\sigma(U_T)\times\sigma(U_T)} \lambda^i\overline{\kappa}^i\ \langle \Pi_\lambda f, g\rangle\overline{\langle \Pi_\kappa f, g\rangle}\ d\nu_T(\lambda)\ d\nu_T(\kappa) \\
&= \int\int_{\sigma(U_T)\times\sigma(U_T)} \frac{1}{n}\sum_{i=0}^{n-1} \lambda^i\overline{\kappa}^i\ \langle \Pi_\lambda f, g\rangle\overline{\langle \Pi_\kappa f, g\rangle}\ d\nu_T(\lambda)\ d\nu_T(\kappa) \\
&= \int\int_{\sigma(U_T)\times\sigma(U_T)} \frac{1}{n}\frac{1 - (\lambda\overline{\kappa})^n}{1 - \lambda\overline{\kappa}}\ \langle \Pi_\lambda f, g\rangle\overline{\langle \Pi_\kappa f, g\rangle}\ d\nu_T(\lambda)\ d\nu_T(\kappa),
\end{aligned}$$

where in the final line we used that the diagonal $\{\lambda = \kappa\}$ has $\nu_T \times \nu_T$-measure zero, because ν is non-atomic (except possibly the atom Π_1 at $\lambda = 1$, but then $\Pi_1 f = 0$). Now $\frac{1}{n}\frac{1-(\lambda\bar{\kappa})^n}{1-\lambda\bar{\kappa}}$ is bounded (use l'Hôpital's rule) and tends to 0 for $\lambda \neq \kappa$, so by the Bounded Convergence Theorem, we have

$$\lim_n \frac{1}{n}\sum_{i=0}^{n-1} |\langle U_T^i f, g\rangle|^2 = 0.$$

Using Corollary 2, we derive that also $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} |\langle U_T^i f, g\rangle| = 0$ (*i.e.*, without the square). Finally, if $f \in L^2(\mu)$ is arbitrary, then $f - \langle f, 1\rangle \in V^\perp$. We find

$$\begin{aligned} 0 &= \lim_n \frac{1}{n}\sum_{i=0}^{n-1} |\langle U_T^i(f - \langle f, 1\rangle), g\rangle| \\ &= \lim_n \frac{1}{n}\sum_{i=0}^{n-1} |\langle U_T^i f - \langle f, 1\rangle, g\rangle| \\ &= \lim_n \frac{1}{n}\sum_{i=0}^{n-1} |\langle U_T^i f, g\rangle - \langle f, 1\rangle\langle 1, g\rangle| \end{aligned}$$

and so property 2. is verified. □

11 Cutting and Stacking

The purpose of **cutting and stacking** is to create invertible maps of the interval that preserve Lebesgue measure, and have further good properties such as "unique ergodicity", "not weak mixing", or rather the opposite "weak mixing but not strong mixing". Famous examples due to Kakutani and to Chacon achieve this, and we will present them here.

The procedure is as follows:

- Cut the unit interval into several intervals, say $A, B, C, \ldots$ (these will become the **stacks**), and a remaining interval S (called the **spacer**).
- Cut each interval into parts (a fix finite number for each stack), and also cut of some intervals from the spacer.
- Pile the parts of the stacks and the cut-off pieces of the spacer on top of the stacks, according to some fixed rule. By choosing the parts in the previous step of the correct size, we can ensure that all intervals in each separate stack have the same size; they can therefore be neatly aligned vertically.

- Map every point on a level of a stack directly to the level above. Then every point has a well-defined image (except for points at the top levels in a stack and points in the remaining spacer), and also a well-defined preimage (except for points at a bottom level in a stack and points in the remaining spacer). Where defined, Lebesgue measure is preserved.

- Repeat the process, now slicing vertically through whole stacks and stacking whole stacks on top of other stacks, possibly putting some intervals of the spacer in between. Wherever the map was defined at a previous step, the definition remains the same.

- Keep repeating. Eventually, the measure of points where the map is not defined tends to zero. In the end, assuming that the spacer will be entirely spent, there will only be one point for each stack without image and one points in each stack without preimage. We can take an arbitrary bijection between them to define the map everywhere.

- The resulting transformation of the interval is invertible and preserves Lebesgue measure. The number of stacks used is called the **rank** of the transformation.

Example 2 (Kakutani)**.** *Take one stack, so start with $A = [0, 1]$. Cut it in half and put the right half on top of the left half. Repeat this procedure. Let us call the result limit map $T : [0, 1] \to [0, 1]$ the Kakutani map. The resulting formula is:*

$$T(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}); \\ x - \frac{1}{4} & \text{if } x \in [\frac{1}{2}, \frac{3}{4}); \\ x - \frac{1}{2} - \frac{1}{8} & \text{if } x \in [\frac{3}{4}, \frac{7}{8}); \\ \quad \vdots & \qquad \vdots \\ x - (1 - \frac{1}{2^n} - \frac{1}{2^{n+1}}) & \text{if } x \in [1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}), n \geqslant 1, \end{cases}$$

see Figure 2. If $x \in [0, 1)$ is written in base 2, i.e.,

$$x = 0.b_1 b_2 b_3 \dots \qquad b_i \in \{0, 1\}, \quad x = \sum_i b_i 2^{-i},$$

then T acts as the **adding machine***: add 0.1 with carry. That is, if $k = \min\{i \geqslant 1 : b_i = 0\}$, then $T(0.b_1 b_2 b_3 \dots) = 0.001 b_{k+1} b_{k+2} \dots$. If $k = \infty$, so $x = 0.111111\dots$, then $T(x) = 0.0000\dots$.*

Proposition 6. *The Kakutani map $T : [0, 1] \to [0, 1]$ of cutting and stacking is uniquely ergodic, but not weakly mixing.*

Proof. The map T permutes the dyadic intervals cyclically. For example $T((0, \frac{1}{2})) = (\frac{1}{2}, 1))$ and $T((\frac{1}{2}, 1)) = (0, \frac{1}{2})$. Therefore, $f(x) = 1_{(0,\frac{1}{2})} - 1_{(\frac{1}{2},1)}$ is an eigenfunction for

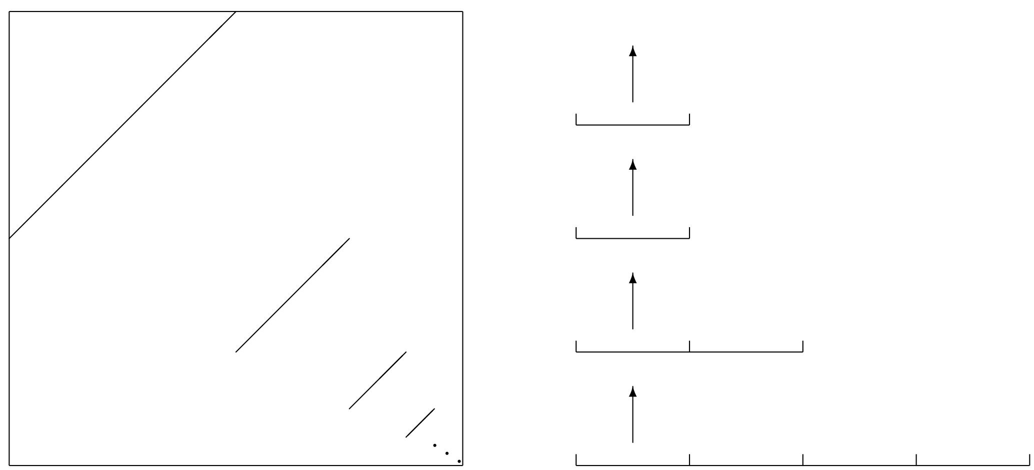

Figure 2: The Kakutani map $T : [0,1] \to [0,1]$ resulting from cutting and stacking.

eigenvalue -1. Using four intervals, we can construct (complex-valued) eigenfunctions for eigenvalues $\pm i$. In generality, all the numbers $e^{2\pi i m/2^n}$, $m, n \in \mathbb{N}$ are eigenvalues, and the corresponding eigenfunctions span $L^2(m)$. This property is called **pure point spectrum**. In any case, T is not weakly mixing.

Now for unique ergodicity, we use the fact again that T permutes the dyadic intervals cyclically. Call these intervals $D_{j,N} = [\frac{j}{2^N}, \frac{j+1}{2^N})$ for $N \in \mathbb{N}$ and $j = \{0, 1, \dots, 2^N - 1\}$, and if $x \in [0,1)$, we indicate the dyadic interval containing it by $D_{j,N}(x)$. Let

$$\begin{cases} \overline{f}_N(x) = \sup_{t \in D_{j,N}(x)} f(t), \\ \underline{f}_N(x) = \inf_{t \in D_{j,N}(x)} f(t), \end{cases}$$

be step-functions that we can use to compute the Riemann integral of f. That is:

$$\int \overline{f}_N(s) ds := \frac{1}{2^N} \sum_{j=0}^{2^N-1} \sup_{t \in D_{j,N}} f(t) \geqslant \int f(s) ds \geqslant \int \underline{f}_N(s) ds := \frac{1}{2^N} \sum_{j=0}^{2^N-1} \inf_{t \in D_{j,N}} f(t).$$

For continuous (or more generally Riemann integrable) functions, $\int \overline{f}_N dx - \int \underline{f}_N dx \to 0$ as $N \to \infty$, and their common limit is called the Riemann integral of f.

According to Lemma 1, we need to show that $\frac{1}{n} \sum_{i=0}^{N-1} f \circ T^i(x)$ converges uniformly to a constant (for each continuous function f) to show that T is uniquely ergodic, *i.e.,* Lebesgue measure is the unique invariant measure.

Let $f : [0,1] \to \mathbb{R}$ be continuous and $\varepsilon > 0$ be given. By uniform continuity, we can find N such that $\max_j(\sup_{t \in D_{j,N}} f(t) - \inf_{t \in D_{j,N}} f(t)) < \varepsilon$. Write $n = m2^N + r$. Any orbit x will visit all intervals $D_{j,N}$ cyclically before returning close to itself, and hence visit each $D_{j,N}$ exactly m times in the first $m2^N$ iterates. Therefore

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} f\circ T^i(x) &\leqslant \frac{1}{m2^N+r}\left(\sum_{j=0}^{2^N-1} m \sup_{t\in D_{j,N}} f(t) + r\|f\|_\infty\right)\\ &\leqslant \frac{1}{2^N}\sum_{j=0}^{2^N-1} \sup_{t\in D_{j,N}} f(t) + \frac{r\|f\|_\infty}{m2^N+r} = \int \overline{f}_N(s)ds + \frac{r\|f\|_\infty}{m2^N+r} \to \int \overline{f}_N(s)ds,\end{aligned}$$

as $m\to\infty$. A similar computation gives $\frac{1}{n}\sum_{i=0}^{n-1} f\circ T^i(x) \geqslant \int \underline{f}_N(x)dx$. As $\varepsilon\to 0$ (and hence $N\to\infty$), we get convergence to the integral $\int f(s)ds$, independently of the initial point x. □

Example 3 (Chacon)**.** *Take one stack and one stack:* $A_0 = [0, \frac{2}{9})$ *and* $S = [\frac{2}{3}, 1)$*. Cut* A_0 *is three equal parts and cut* $[\frac{2}{3}, \frac{8}{9})$ *from spacer* S*. Pile the middle interval* $[\frac{2}{9}, \frac{4}{9})$ *on the left, then the cut-off piece* $[\frac{2}{3}, \frac{8}{9})$ *of the spacer, and then remaining interval* $[\frac{4}{9}, \frac{2}{3})$*. The stack can now be coded upward as* $A_1 = A_0A_0SA_0$*.*

Repeat this procedure: cut the stack vertically in three stacks (of width $\frac{2}{27}$*), cut an interval* $[\frac{8}{9}, \frac{26}{27})$ *from the spacer, and pile them on top of one another: middle stack on left, then the cut-off piece of the spacer, and then the remaining third of the stack. The stack can now be coded upward as* $A_2 = A_1A_1SA_1$*.*

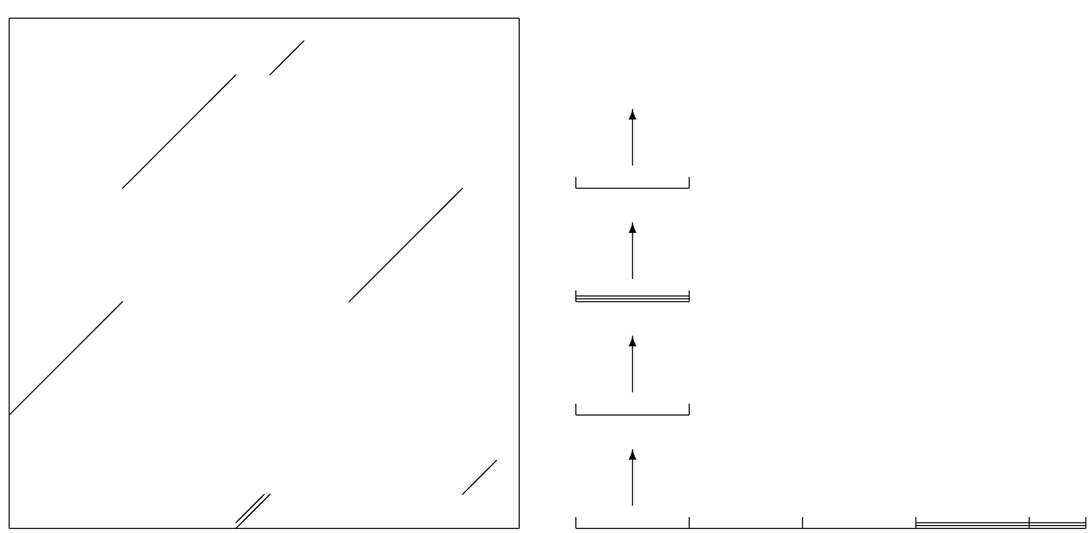

Figure 3: The Chacon map $T : [0,1] \to [0,1]$ resulting from cutting and stacking.

Proposition 7. *The Chacon map* $T : [0,1] \to [0,1]$ *of cutting and stacking is uniquely ergodic, weakly mixing but not strongly mixing.*

Sketch of Proof. First some observations on the symbolic pattern that emerges of the Chacon cutting and stacking. When stacking intervals, their labels follow the following pattern

$$\underbrace{\underbrace{\underbrace{A_0A_0SA_0}_{A_1}\underbrace{A_0A_0SA_0}_{A_1}\,S\,\underbrace{A_0A_0SA_0}_{A_1}}_{A_2}\underbrace{\underbrace{A_0A_0SA_0}_{A_1}\underbrace{A_0A_0SA_0}_{A_1}\,S\,\underbrace{A_0A_0SA_0}_{A_1}}_{A_2}\,S\,\underbrace{\underbrace{A_0A_0SA_0}_{A_1}\underbrace{A_0A_0SA_0}_{A_1}\,S\,\underbrace{A_0A_0SA_0}_{A_1}}_{A_2}}_{A_3}$$

This pattern is the same at every level; we could have started with A_n, grouped together as $A_{n+1} = A_nA_nSA_n$, etc. At step n in the construction of the tower, the width of the stack is $w_n = \frac{2}{3}(3^{-(n+1)}$ and the length of the the word A_n is $l_n = \frac{1}{2}(3^{n+1} - 1)$.

The frequency of each block $\sigma^k(A_n)$ is almost the same in every block huge block B, regardless where taken in the infinite string. This observation leads to unique ergodicity (similar although a bit more involved as in the case of the Kakutani map), but we will skip the details.

Instead, we focus on the weak mixing. Clearly the word A_n appears in triples, and also as $A_nA_nA_nSA_nA_nA_n$. To explain the idea behind the proof, pretend that an eigenfunction (with eigenvalue $e^{2\pi i\lambda}$) were constant on any set E whose code is A_n (or $\sigma^k A_n$ for some $0 \leqslant k < l_n$, where σ denotes the left-shift). Such set E are intervals of width w_n. Then

$$f \circ T^{l_n}|_E = e^{2\pi i\lambda l_n} f|_E \text{ and } f \circ T^{2l_n+1}|_E = e^{2\pi i\lambda l_n} f|_E$$

This gives $1 = e^{2\pi i\lambda l_n} = e^{2\pi i\lambda l_n}$, so $\lambda = 0$, and the eigenvalue is 1 after all.

The rigorous argument is as follows. Suppose that $f(x) = e^{2\pi i\vartheta(x)}$ were an eigenfunction for eigenvalue $e^{2\pi i\lambda}$ and a measurable function $\vartheta : \mathbb{S}^1 \to \mathbb{R}$. By Lusin's Theorem, we can find a subset $F \subset \mathbb{S}^1$ of Lebesgue measure $\geqslant 1 - \varepsilon$ such that ϑ is uniformly continuous on F. Choose $\varepsilon > 0$ arbitrary, and take N so large that the variation of ϑ is less that ε on any set of the form $E \cap F$, where points in E have code starting as $\sigma^k(A_N)$, $0 \leqslant k < l_N$. Sets of this type fill a set E^* with mass at least half of the unit interval.

Because of the frequent occurrence of $A_NA_NA_NSA_NA_NA_N$, a definite proportion of E^* is covered by set E with the property that such that $T^{2l_N+1} \cap T^{l_N}E \cap E \neq \varnothing$, because they have codes of length l_N that reappear after both l_N and $2l_N + 1$ shifts. For x in this intersection,

$$\begin{cases} \vartheta \circ T^{2l_N+1}(x) = (l_N + 1)\lambda + \vartheta \circ T^{l_N}(x) \pmod 1 \\ \vartheta \circ T^{l_N}(x) = l_N\lambda + \vartheta(x) \pmod 1 \end{cases}$$

where all three point $x, T^{l_N}(x), T^{2l_N+1}(x)$ belong to the same copy E. Subtracting the two equations gives

$$\lambda \bmod 1 = \vartheta \circ T^{2l_N+1}(x) - \vartheta \circ T^{l_N}(x) + \vartheta(x) - \vartheta \circ T^{l_N}(x) \leqslant 2\varepsilon.$$

But ε is arbitrary, so $\lambda = 0 \bmod 1$ and the eigenvalue is 1.

Now for the strong mixing, consider once more the sets $E = E_{k,n}$ of points whose codes starts as the k-th cyclic permutation of A_n for some $0 \leqslant k < l_n$, that is: the first l_n symbols of $\sigma^k(A_n A_n)$. Their measure is $\mu(E) = w_n$, and for different k, they are disjoint. Furthermore, the only l_n-block appearing are cyclic permutations of A_n or cyclic permutations with a spacer S inserted somewhere. At least half of these appearances are of the first type, so $\mu(\cup_{k=0}^{l_n-1} E_{k,n}) \geqslant \frac{1}{2}$ for each n.

The basic idea is that $\mu(E \cap T^{-l_n} E) \geqslant \frac{1}{3}\mu(E)$ because at least a third of the appearances of A_n is followed by another A_n. But $\frac{1}{3}\mu(E) \gg \mu(E)^2$, as one would expect for mixing. Of course, mixing only says that $\lim_l \mu(Y \cap T^{-l}(E)) = \mu(Y)^2$ only for sets Y not depending on l.

However, let $Y_m = [m/8, (m+1)/8] \subset [0,1]$, $m = 0, \dots, 7$ be the eight dyadic intervals of length 1/8. For each n, at least one Y_m is covered for at least half by sets E of the above type, say a set $Z \subset Y_m$ of measure $\mu(Z) \geqslant \frac{1}{2}\mu(Y_m))$ such that $Z \subset \cup_k E_{k,n}$. That means that

$$\mu(Y_m \cap T^{-l_n}(Y_m)) \geqslant \mu(Z \cap T^{-l_n}(Z)) \geqslant \frac{1}{3}\mu(Z) \geqslant \frac{1}{6}\mu(Y_m) > \mu(Y_m)^2.$$

Let Y be one of the Y_m's for which the above holds for infinitely many n. Then $\limsup_n \mu(Y_m \cap T^{-l_n}(Y_m)) > \mu(Y)^2$, contradicting strong mixing. □

12 Toral automorphisms

The best known example of a toral automorphism (that is, an invertible linear map on the torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$) is the Arnol'd cat map. This map $T_C : \mathbb{T}^2 \to \mathbb{T}^2$ is defined as

$$T_C(x,y) = C \begin{pmatrix} x \\ y \end{pmatrix} \pmod 1 \qquad \text{for the matrix } C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The name come from the illustration in Arnol'd's book [3] showing how the head of a cat, drawn on a torus, is distorted by the action of the map[2]. Properties of T_C are:

- C preserves the integer lattice, so T_C is well-defined an continuous.
- $\det(C) = 1$, so Lebesgue measure m is preserved (both by C and T_C). Also C and T_C are invertible, and C^{-1} is still an integer matrix.
- The eigenvalues of C are $\lambda_\pm = (3 \pm \sqrt{5})/2$, and the corresponding eigenspaces $E_\pm$ are spanned $(-1, (\sqrt{5}+1)/2)^T$ and $(1, (\sqrt{5}-1)/2)^T$. These are orthogonal

[2]Arnol'd didn't seem to like cats, but see the applet https://www.jasondavies.com/catmap/ how the cat survives

(naturally, since C is symmetric), and have irrational slopes, so they wrap densely in the torus.

- Every rational point in $\mathbb{T}^2$ is periodic under T (as their denominators cannot increase, so T acts here as an invertible map on a finite set). This gives many invaraint measures: the equidistribution on each periodic orbit. Therefore T_C is not uniquely ergodic.

The properties are common to all maps T_A, provided they satisfy the following definition.

Definition 11. *A toral automorphism $T : \mathbb{T}^d \to \mathbb{T}^d$ is an invertible linear map on the (d-dimensional) torus $\mathbb{T}^d$. Each such T is of the form $T_A(x) = Ax \pmod 1$, where the matrix A satisfies:*

- *A is an integer matrix with $\det(A) = \pm 1$;*
- *the eigenvalues of A are not on the unit circle; this property is called* **hyperbolicity**.

Somewhat easier to treat that the cat map is T_A for $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which is an orientation reversing matrix with $A^2 = C$. The map T_A has a **Markov partition**, that is a partition $\{R_i\}_{i=1}^N$ for sets such that

1. The R_i have disjoint interiors and $\cup_i R_i = \mathbb{T}^d$;
2. If $T_A(R_i) \cap R_j \neq \varnothing$, then $T_A(R_i)$ stretches across R_j in the unstable direction (*i.e.,* the direction spanned by the unstable eigenspaces of A).
3. If $T_A^{-1}(R_i) \cap R_j \neq \varnothing$, then $T_A^{-1}(R_i)$ stretches across R_j in the stable direction (*i.e.,* the direction spanned by the stable eigenspaces of A).

In fact, every hyperbolic toral automorphism has a Markov partition, but in general they are fiendishly difficult to find explicitly. In the case of A, a Markov partition of three rectangles R_i for $i = 1, 2, 3$ can be constructed, see Figure 4.

The corresponding transition matrix is

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{where } B_{ij} = \begin{cases} 1 & \text{if } T_A(R_i) \cap R_j \neq \varnothing \\ 0 & \text{if } T_A(R_i) \cap R_j = \varnothing. \end{cases}$$

Note that the characteristic polynomial of B is

$$\det(B - \lambda I) = -\lambda^3 + 2\lambda + 1 = -(\lambda + 1)(\lambda^2 - \lambda - 1) = -(\lambda + 1) \det(A - \lambda I).$$

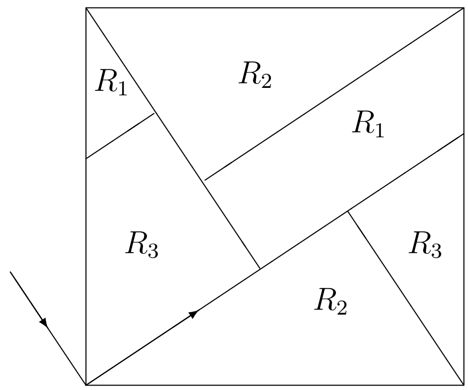

Figure 4: The Markov partition for the toral automorphism T_A. The arrows indicate the stable and unstable directions at $(0,0)$.

so B has the eigenvalues of A (no coincidence!), together with $\lambda = -1$. The transition matrix B generates a **subshift of finite type**:

$$\Sigma_B = \{(x_i)_{i\in\mathbb{Z}} : x_i \in \{1,2,3\}, B_{x_i x_{i+1}} = 1 \ \forall\ i \in \mathbb{Z}\},$$

equipped with the left-shift σ. That is, Σ_B contains only sequences in which each $x_i x_{i+1}$ indicate transitions from Markov partition elements that are allowed by the map T_A.

It can be shown that $(\mathbb{T}^d, \mathcal{B}, T, Leb)$ is isomorphic to the shift space $(\Sigma_B, \mathcal{C}, \sigma, \mu)$ where

$$\mu([x_k x_{k+1} \dots x_n]) = m_{x_k} \Pi_{x_k x_{k+1}} \Pi_{x_{k+1} x_{k+2}} \dots \Pi_{x_{n-1} x_n},$$

for $m_i = Leb(R_i)$, $i = 1, \dots, d\}$, and weighted transition matrix Π where

$$\Pi_{ij} = \frac{Leb(T_A(R_i) \cap R_j)}{Leb(R_i)} \text{ is the relative mass that } T_A \text{ transports from } R_i \text{ to } R_j.$$

Finally $\mathcal{C}$ the σ-algebra of set generated by allowed cylinder sets.

Theorem 12. *For every hyperbolic toral automorphism, Lebesgue measure is ergodic and mixing.*

Proof. We only give the proof for dimension 2. The higher dimensional case goes similarly. Consider the Fourier modes (also called **characters**)

$$\chi_{(m,n)} : \mathbb{T}^2 \to \mathbb{C}, \qquad \chi_{(m,n)}(x,y) = e^{2\pi i(mx+ny)}.$$

These form an orthogonal system (w.r.t. $\langle \varphi, \psi \rangle = \int \varphi \overline{\psi} d\lambda$), spanning $L^2(\lambda)$ for Lebesgue measure λ. We have

$$U_{T_A} \chi_{(m,n)}(x,y) = \chi_{(m,n)} \circ T_A(x,y) = \chi_{m,n}(x,y) = e^{2\pi i((am+cn)x+(bm+dn)y)} = \chi_{A^t(m,n)}(x,y).$$

In other words, U_{T_A} maps the character with index (m, n) to the character with index $A^t(m, n)$, where A^t is the transpose matrix.

For the proof of ergodicity, assume that φ is a T_A-invariant L^2-function. Write it as Fourier series:

$$\varphi(x, y) = \sum_{m,n\in\mathbb{Z}} \varphi_{(m,n)}\chi_{(m,n)}(x, y),$$

where the Fourier coefficients $\varphi_{m,n} \to 0$ as $|m| + |n| \to \infty$ By T_A-invariance, we have

$$\varphi(x, y) = \varphi \circ T_A(x, y) = \sum_{m,n\in\mathbb{Z}} \varphi_{(m,n)}\chi_{A^t(m,n)}(x, y),$$

and hence $\varphi_{(m,n)} = \varphi_{A^t(m,n)}$ for all m, n. For $(m, n) = (0, 0)$ this is not a problem, but this only produces constant functions. If $(m, n) \neq (0, 0)$, then the A^t-orbit of (m, n), so infinitely many equal Fourier coefficients

$$\varphi_{(m,n)} = \varphi_{A^t(m,n)} = \varphi_{(A^t)^2(m,n)} = \varphi_{(A^t)^3(m,n)} = \varphi_{(A^t)^4(m,n)} \cdots$$

As the Fourier coefficients converge to zero as $|m| + |n| \to \infty$, they all must be equal to zero, and hence φ is a constant function. This proves ergodicity.

For the proof of mixing, we need a lemma, which we give without proof.

Lemma 6. *A transformation (X, T, μ) is mixing if and only if for all φ, ψ in a complete orthogonal system spanning $L^2(\mu)$, we have*

$$\int_X \varphi \circ T^N(x)\overline{\psi(x)}\, d\mu \to \int_X \varphi(x)\, d\mu \cdot \int_X \overline{\psi(x)}\, d\mu$$

as $N \to \infty$.

To use this lemma on $\varphi = \chi_{(m,n)}$ and $\psi = \chi_{(k,l)}$, we compute

$$\int_X \chi_{(m,n)} \circ T^N(x)\overline{\chi_{(k,l)}(x)}\, d\lambda = \int_X \chi_{(A^t)^N(m,n)}\overline{\chi_{(k,l)}(x)}\, d\lambda.$$

If $(m, n) = (0, 0)$, then $(A^t)^N(m, n) = (0, 0) = (m, n)$ for all N. Hence, the integral is non-zero only if $(k, l) = (0, 0)$, but then the integral equals 1, which is the same as $\int_X \chi_{(0,0)}\, d\lambda \int_X \overline{\chi_{(0,0)}(x)}\, d\lambda$. If $(k, l) = (0, 0)$, then the integral is zero, but so is $\int_X \chi_{(0,0)}\, d\lambda \int_X \overline{\chi_{(0,0)}(x)}\, d\lambda$.

If $(m, n) \neq (0, 0)$, then, regardless what (k, l) is, there is N such that $(A^t)^M(m, n) \neq (k, l)$ for all $M \geqslant N$. Therefore

$$\int_X \chi_{(m,n)} \circ T^M(x)\overline{\chi_{(k,l)}(x)}\, d\lambda = 0 = \int_X \chi_{(m,n)}\, d\lambda \int_X \overline{\chi_{(k,l)}(x)}\, d\lambda.$$

The lemma therefore guarantees mixing. □

13 Topological entropy and topological pressure

Topological entropy was first defined in 1965 by Adler et al. [1], but the form that Bowen [4] and Dinaburg [8] redressed it in is commonly used nowadays.

We will start by start giving the original definition, because the idea of joints of covers easily relates to joints of partitions as used in measure-theoretic entropy. After that, we will give Bowen's approach, since it readily generalises to topological pressure as well.

13.1 The orginal definition

Let (X, d, T) be a continuous map on compact metric space (X, d). We say that $\mathcal{U} = \{U_i\}$ is an *open ε-cover* if all U_i are open sets of diamter $\leqslant \varepsilon$ and $X \subset \bigcup_i U_i$. Naturally, compactness of X guarantees that for every open cover, we can select a finite subcover. Thus, let $\mathcal{N}(\mathcal{U})$ the the minimal possible cardinality of subcovers of $\mathcal{U}$. We say that $\mathcal{U}$ *refines* $\mathcal{V}$ (notation $\mathcal{U} \succeq \mathcal{V}$) if evey $U \in \mathcal{U}$ is contained in a $V \in \mathcal{V}$. If $\mathcal{U} \succeq \mathcal{V}$ then $\mathcal{N}(\mathcal{U}) \geq \mathcal{N}(\mathcal{V})$.

Given two cover $\mathcal{U}$ and $\mathcal{V}$, the joint

$$\mathcal{U} \vee \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$$

is an open cover again, and one can verify that $\mathcal{N}(\mathcal{U} \vee \mathcal{V}) \leqslant \mathcal{N}(\mathcal{U})\mathcal{N}(\mathcal{V})$. Since T is continuous, $T^{-1}(\mathcal{U})$ is an open cover as well, although in this case it need not be an ε-cover; However, $\mathcal{U} \vee T^{-1}(\mathcal{U})$ is an ε-cover, and it refines $T^{-1}(\mathcal{U})$.

Define the *topological entropy* as

$$h_{top}(T) = \lim_{\varepsilon \to 0} \sup_{\mathcal{U}} \lim_{n} \frac{1}{n} \log \mathcal{N}(\mathcal{U}^n) \qquad \text{for } \mathcal{U}^n := \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U}), \tag{10}$$

where the supremum is taken over all open ε-covers $\mathcal{U}$. Because $\mathcal{N}(\mathcal{U} \vee \mathcal{V}) \leqslant \mathcal{N}(\mathcal{U})\mathcal{N}(\mathcal{V})$, the sequence $\log \mathcal{N}(\mathcal{U}^n)$ is subadditive, so the limit $\lim_n \frac{1}{n} \log \mathcal{N}(\mathcal{U}^n)$ exists. We have the following properties:

Lemma 7.

- *$h_{top}(T^k) = k h_{top}(T)$ for $k \geqslant 0$. If T is invertible, then also $h_{top}(T^{-1}) = h_{top}(T)$.*
- *If (Y, S) is semiconjugate to (X, T), then $h_{top}(S) \leqslant h_{top}(T)$. In particular, conjugate systems (on compact spaces!) have the same entropy.*

Proof. □

13.2 Topological entropy of interval maps

If $X = [0,1]$ with the usual Euclidean metric, then there are various shortcuts to compute the entropy of a continuous map $T : [0,1] \to [0,1]$. Let us call any maximal interval on which T is monotone a *lap*; the number of laps is denoted as $\ell(T)$. Also, the *variation* of T is defined as

$$Var(T) = \sup_{0 \leqslant x_0 < \dots\ x_N \leqslant N} \sum_{i=1}^{N} |T(x_i) - T(x_{i-1})|,$$

where the supremum runs over all finite collections of points in $[0,1]$. The following result is due to Misurewicz & Szlenk [12].

Proposition 8. *Let $T : [0,1] \to [0,1]$ have finitely many laps. Then*

$$\begin{aligned} h_{top}(T) &= \lim_{n\to\infty} \frac{1}{n} \log \ell(T^n) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \#\{\textit{clusters of } n\textit{-periodic points}\} \\ &= \max\{0, \lim_{n\to\infty} \frac{1}{n} \log Var(T^n)\}. \end{aligned}$$

where two n-periodic points are in the same cluster if they belong to the same lap of T^n.

Proof. Since the variation of a monotone function is given by $\sup T - \inf T$, and due to the definition of "cluster" of n-periodic points, the inequalities

$$\#\{\text{clusters of } n\text{-periodic points}\}, \mathrm{Var}(T^n) \leqslant \ell(T^n)$$

are immediate. For a lap I of T^n, let $\gamma := |T^n(I)|$ be its *height*. We state without proof (cf. [5, Chapter 9]):

$$\begin{array}{l} \text{For every } \delta > 0 \text{, there is } \gamma > 0 \text{ such that the number of} \\ \text{laps } I \text{ of } T^n \text{ with the property that } I \text{ belongs to a lap of } T^j \\ \text{of height} \geqslant \gamma \text{ for all } 1 \leqslant j \leqslant n \text{ is at least } (1-\delta)^n \ell(T^n). \end{array} \tag{11}$$

This means that $Var(T^n) \geqslant \gamma(1-\delta)^n \ell(T^n)$, and therefore

$$-2\delta + \lim_n \frac{1}{n} \log \ell(T^n) \leqslant \lim_n \frac{1}{n} \log \mathrm{Var}(T^n) \leqslant \lim_n \frac{1}{n} \log \ell(T^n).$$

Since δ is arbitrary, both above quantities are all equal.

Making the further assumption (without proof[3]) that there is $K = K(\gamma)$ such that $\cup_{i=0}^{K} T^i(J) = X$ for every interval of length $|J| \geqslant \gamma$, we also find that

$$\#\{\text{clusters of } n+i\text{-periodic points}, 0 \leqslant i \leqslant K\} \geqslant (1-\delta)^n \ell(T^n).$$

[3] In fact, it is not entirely true if T has an invariant subset attracting an open neighbourhood. But it suffices to restrict T to its *nonwandering set*, that is, the set $\Omega(T) = \{x \in X : x \in \cup_{n \geqslant 1} T^n(U))$ for every neighbourhood $U \ni x\}$, because $h_{top}(T) = h_{top}(T|_{\Omega(T)})$.

This implies that

$$-2\delta + \lim_n \frac{1}{n} \log \ell(T^n) \leqslant \limsup_n \frac{1}{n} \max_{0 \leqslant i \leqslant K} \log \#\{\text{clusters of } n+i\text{-periodic points}\}$$

so also $\lim_n \frac{1}{n} \log \ell(T^n) = \limsup_{n\to\infty} \frac{1}{n} \log \#\{\text{clusters of } n\text{-periodic points}\}$

If $\varepsilon > 0$ is so small that the width of every lap is greater than 2ε, then for every ε-cover $\mathcal{U}$, every subcover of $\mathcal{U}^n$ has at least one element in each lap of T^n. Therefore $\ell(T^n) \leqslant \mathcal{N}(\mathcal{U}^n)$ for every ε-cover, so $\lim_n \frac{1}{n} \log \ell(T^n) \leqslant h_{top}(T)$. □

13.3 Bowen's approach

Let T be map of a compact metric space (X, d). If my eyesight is not so good, I cannot distinguish two points $x, y \in X$ if they are at a distance $d(x, y) < \varepsilon$ from one another. I may still be able to distinguish there orbits, if $d(T^k x, T^k y) > \varepsilon$ for some $k \geqslant 0$. Hence, if I'm willing to wait $n-1$ iterations, I can distinguish x and y if

$$d_n(x, y) := \max\{d(T^k x, T^k y) : 0 \leqslant k < n\} > \varepsilon.$$

If this holds, then x and y are said to be (n, ε)**-separated**. Among all the subsets of X of which all points are mutually (n, ε)-separated, choose one, say $E_n(\varepsilon)$, of maximal cardinality. Then $s_n(\varepsilon) := \#E_n(\varepsilon)$ is the maximal number of n-orbits I can distinguish with ε-poor eyesight.

The **topological entropy** is defined as the limit (as $\varepsilon \to 0$) of the exponential growth-rate of $s_n(\varepsilon)$:

$$h_{top}(T) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon). \tag{12}$$

Note that $s_n(\varepsilon_1) \geqslant s_n(\varepsilon_2)$ if $\varepsilon_1 \leqslant \varepsilon_2$, so $\limsup_n \frac{1}{n} \log s_n(\varepsilon)$ is a decreasing function in ε, and the limit as $\varepsilon \to 0$ indeed exists.

Instead of (n, ε)-separated sets, we can also work with (n, ε)**-spanning** sets, that is, sets that contain, for every $x \in X$, a y such that $d_n(x, y) \leqslant \varepsilon$. Note that, due to its maximality, $E_n(\varepsilon)$ is always (n, ε)-spanning, and no proper subset of $E_n(\varepsilon)$ is (n, ε)-spanning. Each $y \in E_n(\varepsilon)$ must have a point of an $(n, \varepsilon/2)$-spanning set within an $\varepsilon/2$-ball (in d_n-metric) around it, and by the triangle inequality, this $\varepsilon/2$-ball is disjoint from $\varepsilon/2$-ball centred around all other points in $E_n(\varepsilon)$. Therefore, if $r_n(\varepsilon)$ denotes the minimal cardinality among all (n, ε)-spanning sets, then

$$r_n(\varepsilon) \leqslant s_n(\varepsilon) \leqslant r_n(\varepsilon/2). \tag{13}$$

Thus we can equally well define

$$h_{top}(T) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon). \tag{14}$$

Examples: Consider the β-transformation $T_\beta : [0,) \to [0,1)$, $x \mapsto \beta x \pmod 1$ for some $\beta > 1$. Take $\varepsilon < 1/(2\beta^2)$, and $G_n = \{\frac{k}{\beta^n} \; : \; 0 \leqslant k < \beta^n\}$. Then G_n is (n,ε)-separating, so $s_n(\varepsilon) \geqslant \beta^n$. On the other hand, $G'_n = \{\frac{2k\varepsilon}{\beta^n} \; : \; 0 \leqslant k < \beta^n/(2\varepsilon)\}$ is (n,ε)-spanning, so $r_n(\varepsilon) \leqslant \beta^n/(2\varepsilon)$. Therefore

$$\log \beta = \limsup_n \frac{1}{n} \log \beta^n \leqslant h_{top}(T_\beta) \leqslant \limsup_n \log \beta^n/(2\varepsilon) = \log \beta.$$

Circle rotations, or in general isometries, T have zero topological entropy. Indeed, if $E(\varepsilon)$ is an ε-separated set (or ε-spanning set), it will also be (n,ε)-separated (or (n,ε)-spanning) for every $n \geqslant 1$. Hence $s_n(\varepsilon)$ and $r_n(\varepsilon)$ are bounded in n, and their exponential growth rates are equal to zero.

Finally, let (X,σ) be the full shifts on N symbols. Let $\varepsilon > 0$ be arbitrary, and take m such that $2^{-m} < \varepsilon$. If we select a point from each $n+m$-cylinder, this gives an (n,ε)-spanning set, whereas selecting a point from each n-cylinder gives an (n,ε)-separated set. Therefore

$$\begin{aligned} \log N = \limsup_n \frac{1}{n} \log N^n &\leqslant \limsup_n \frac{1}{n} \log s_n(\varepsilon) \leqslant h_{top}(T_\beta) \\ &\leqslant \limsup_n \frac{1}{n} \log r_n(\varepsilon) \leqslant \limsup_n \log N^{n+m} = \log N. \end{aligned}$$

Proposition 9. *For a continuous map T on a compact metric space (X,d), the three definitions* (10)*,* (12) *and* (14) *give the same outcome.*

Proof. The equality of the limits (12) and (14) follows directly from (13).

If $\mathcal{U}$ is an ε-cover, every $A \in \mathcal{U}^n$ can contain at most one point in an (n,ε)-separated set, so $s(n,\varepsilon) < \mathcal{N}(\mathcal{U}^n)$, whence $\limsup_n \frac{1}{n} \log s(n,\varepsilon) \leqslant \lim_n \frac{1}{n} \log \mathcal{N}(\mathcal{U}^n)$.

Finally, in a compact metric space, every open cover $\mathcal{U}$ has a numbr (called its *Lebesgue number*) such that for every $x \in X$, there is $U \imath \mathcal{U}$ such that $B_\delta(x) \subset U$. Clearly $\delta < \varepsilon$ if $\mathcal{U}$ is an ε-cover. Now if an open ε-cover $\mathcal{U}$ has Lebesgue number δ, and E is an $(n,\delta$-spanning set of cardinality $\#E = r(n,\delta)$, then $X \subset \cup_{x \in E}$ $cap_{i=0}^{n-1} T^{-i}(B_\delta(T^i))$. Since each $B_\delta(T^i(x))$ is contained in some $U \in \mathcal{U}$, we have $\mathcal{N}(\mathcal{U}^n) \leqslant r(n,\delta)$. Since $\delta \to 0$ as $\varepsilon \to 0$, also

$$\lim_{\varepsilon \to 0} \lim_n \frac{1}{n} \log \mathcal{N}(\mathcal{U}^n) \leqslant \lim_{\delta \to 0} \limsup_n \frac{1}{n} \log r(n,\delta).$$

□

13.4 Topological pressure

The topological pressure $P_{top}(T,\psi)$ combines entropy with a potential function $\psi : X \to \mathbb{R}$. By definition, $h_{top}(T) = P_{top}(T,\psi)$ if $\psi(x) \equiv 0$. Denote the n-th ergodic sum of ψ

by

$$S_n\psi(x) = \sum_{k=0}^{n-1} \psi \circ T^k(x).$$

Next set

$$\begin{cases} K_n(T,\psi,\varepsilon) = \sup\{\sum_{x\in E} e^{S_n\psi(x)} \ : \ E \text{ is } (n,\varepsilon)\text{-separated}\}, \\ L_n(T,\psi,\varepsilon) = \inf\{\sum_{x\in E} e^{S_n\psi(x)} \ : \ E \text{ is } (n,\varepsilon)\text{-spanning}\}. \end{cases} \tag{15}$$

For reasonable choices of potentials, the quantities $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log K_n(T,\psi,\varepsilon)$ and $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log L_n(T,\psi,\varepsilon)$ are the same, and this quantity is called the **topological pressure**. To give an example of an unreasonable potential, take X_0 be a dense T-invariant subset of X such that $X \setminus X_0$ is also dense. Let

$$\psi(x) = \begin{cases} 100 & \text{if } x \in X_0, \\ 0 & \text{if } x \notin X_0. \end{cases}$$

Then $L_n(T,\psi,\varepsilon) = r_n(\varepsilon)$ whilst $K_n(T,\psi,\varepsilon) = e^{100n}s_n(\varepsilon)$, and their exponential growth rates differ by a factor 100. Hence, some amount of continuity of ψ is necessary to make it work.

Lemma 8. *If $\varepsilon > 0$ is such that $d(x,y) < \varepsilon$ implies that $|\psi(x) - \psi(y)| < \delta/2$, then*

$$e^{-n\delta}K_n(T,\psi,\varepsilon) \leqslant L_n(T,\psi,\varepsilon/2) \leqslant K_n(T,\psi,\varepsilon/2).$$

Exercise 2. *Prove Lemma 8. In fact, the second inequality holds regardless of what ψ is.*

Theorem 13. *If $T : X \to X$ and $\psi : X \to \mathbb{R}$ are continuous on a compact metric space, then the topological pressure is well-defined by*

$$P_{top}(T,\psi) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log K_n(T,\psi,\varepsilon) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log L_n(T,\psi,\varepsilon).$$

Exercise 3. *Show that $P_{top}(T^R, S_R\psi) = R \cdot P_{top}(T,\psi)$.*

14 Measure-theoretic entropy

Entropy is a measure for the complexity of a dynamical system (X,T). In the previous sections, we related this (or rather topological entropy) to the exponential growth rate of the cardinality of $\mathcal{P}_n = \bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}$ for some partition of the space X. In this section, we look at the measure theoretic entropy $h_\mu(T)$ of an T-invariant measure μ, and this amounts to, instead of just counting $\mathcal{P}_n$, taking a particular weighted sum of the elements $Z_n \in \mathcal{P}_n$. However, if the mass of μ is equally distributed over the all the $Z_n \in$

$\mathcal{P}_n$, then the outcome of this sum is largest; then μ would be the measure of maximal entropy. In "good" systems (X,T) is indeed the supremum over the measure theoretic entropies of all the T-invariant probability measures. This is called the **Variational Principle**:

$$h_{top}(T) = \sup\{h_\mu(T) \ : \ \mu \text{ is } T\text{-invariant probability measure}\}. \tag{16}$$

In this section, we will skip some of the more technical aspect, such as **conditional entropy** (however, see Proposition 10) and σ-algebras (completing a set of partitions), and this means that at some points we cannot give full proofs. Rather than presenting more philosophy what entropy should signify, let us first give the mathematical definition.

Define

$$\varphi : [0,1] \to \mathbb{R} \quad \varphi(x) = -x \log x$$

with $\varphi(0) := \lim_{x\downarrow 0} \varphi(x) = 0$. Clearly $\varphi'(x) = -(1+\log x)$ so $\varphi(x)$ assume its maximum at $1/e$ and $\varphi(1/e) = 1/e$. Also $\varphi''(x) = -1/x < 0$, so that φ is **strictly concave**:

$$\alpha\varphi(x) + \beta\varphi(y) \leqslant \varphi(\alpha x + \beta y) \qquad \text{for all } \alpha+\beta = 1, \alpha, \beta \geqslant 0, \tag{17}$$

with equality if and only if $x = y$.

Theorem 14 (Jensen's Inequality)**.** *For every strictly concave function $\varphi : [0,\infty) \to \mathbb{R}$ we have*

$$\sum_i \alpha_i \varphi(x_i) \leqslant \varphi(\sum_i \alpha_i x_i) \text{ for } \alpha_i > 0, \sum_i \alpha_i = 1 \text{ and } x_i \in [0,\infty), \tag{18}$$

with equality if and only if all the x_i are the same.

Proof. We prove this by induction on n. For $n = 2$ it is simply (17). So assume that (18) holds for some n, and we treat the case $n+1$. Assume $\alpha_i > 0$ and $\sum_{i=1}^{n+1} \alpha_i = 1$ and write $B = \sum_{i=1}^n \alpha_i$.

$$\begin{aligned}
\varphi(\sum_{i=1}^{n+1} \alpha_i x_i) &= \varphi(B \sum_{i=1}^{n} \frac{\alpha_i}{B} x_i + \alpha_{n+1} x_{n+1}) \\
&\geqslant B\varphi(\sum_{i=1}^{n} \frac{\alpha_i}{B} x_i) + \varphi(\alpha_{n+1} x_{n+1}) \qquad \text{by (17)} \\
&\geqslant B \sum_{i=1}^{n} \frac{\alpha_i}{B} \varphi(x_i) + \varphi(\alpha_{n+1} x_{n+1}) \qquad \text{by (18) for } n \\
&= \sum_{i=1}^{n+1} \alpha_i \varphi(x_i)
\end{aligned}$$

as required. Equality also carries over by induction, because if x_i are all equal for $1 \leqslant i \leqslant n$, (17) only preserves equality if $x_{n+1} = \sum_{i=1}^n \frac{\alpha_i}{B} x_i = x_1$. □

This proof doesn't use the specific form of φ, only its (strict) concavity. Applying it to $\varphi(x) = -x \log x$, we obtain:

Corollary 3. *For $p_1 + \cdots + p_n = 1$, $p_i > 0$, then $\sum_{i=1}^n \varphi(p_i) \leqslant \log n$ with equality if and only if all p_i are equal,* i.e., $p_i \equiv \frac{1}{n}$.

Proof. Take $\alpha_i = \frac{1}{n}$, then by Theorem 14,

$$\frac{1}{n}\sum_{i=1}^n \varphi(p_i) = \sum_{i=1}^n \alpha_i \varphi(p_i) \leqslant \varphi(\sum_{i=1}^n \frac{1}{n} p_i) = \varphi(\frac{1}{n}) = \frac{1}{n}\log n.$$

Now multiply by n. $\square$

Corollary 4. *For real numbers a_i and $p_1 + \cdots + p_n = 1$, $p_i > 0$, $\sum_{i=1}^n p_i(a_i - \log p_i) \leqslant \log \sum_{i=1}^n e^{a_i}$ with equality if and only if $p_i = e^{a_i}/\sum_{i=1}^n e^{a_i}$ for each i.*

Proof. Write $\mathcal{Z} = \sum_{i=1}^n e^{a_i}$. Put $\alpha_i = e^{a_i}/\mathcal{Z}$ and $x_i = p_i\mathcal{Z}/e^{a_i}$ in Theorem 14. Then

$$\begin{aligned} \sum_{i=1}^n p_i(a_i - \log \mathcal{Z} - \log p_i) &= -\sum_{i=1}^n \frac{e^{a_i}}{\mathcal{Z}} \left(\frac{p_i \mathcal{Z}}{e^{a_i}} \log \frac{p_i \mathcal{Z}}{e^{a_i}} \right) \\ &\leqslant -\sum_{i=1}^n \frac{e^{a_i}}{\mathcal{Z}} \frac{p_i \mathcal{Z}}{e^{a_i}} \log \sum_{i=1}^n \frac{e^{a_i}}{\mathcal{Z}} \frac{p_i \mathcal{Z}}{e^{a_i}} = \varphi(1) = 0. \end{aligned}$$

Rearranging gives $\sum_{i=1}^n p_i(a_i - \log p_i) \leqslant \log Z$, with equality only if $x_i = p_i Z/e^{a_i}$ are all the same, *i.e.*, $p_i = e^{a_i}/Z$. $\square$

Exercise 4. *Reprove Corollaries 3 and 4 using Lagrange multipliers.*

Given a finite partition $\mathcal{P}$ of a probability space (X, μ), let

$$H_\mu(\mathcal{P}) = \sum_{P \in \mathcal{P}} \varphi(\mu(P)) = -\sum_{P \in \mathcal{P}} \mu(P) \log(\mu(P)), \tag{19}$$

where we can ignore the partition elements with $\mu(P) = 0$ because $\varphi(0) = 0$. For a T-invariant probability measure μ on $(X, \mathcal{B}, T)$, and a partition $\mathcal{P}$, define the **entropy of μ w.r.t. $\mathcal{P}$** as

$$H_\mu(T, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}). \tag{20}$$

Finally, the **measure theoretic entropy** of μ is

$$h_\mu(T) = \sup\{H_\mu(T, \mathcal{P}) \ : \ \mathcal{P} \text{ is a finite partition of } X\}. \tag{21}$$

Naturally, this raises the questions:

Does the limit exist in (20)?
How can one possibly consider **all** partitions of X?

We come to this later; first we want to argue that entropy is a characteristic of a measure preserving system. That is, two measure preserving systems $(X, \mathcal{B}, T, \mu)$ and $(Y, \mathcal{C}, S, \nu)$ that are **isomorphic**, *i.e.,* there are full-measured sets $X_0 \subset X$, $Y_0 \subset Y$ and a bi-measurable invertible measure-preserving map $\pi : X_0 \to Y_0$ (called **isomorphism**) such that the diagram

$$\begin{array}{ccc} (X_0, \mathcal{B}, \mu) & \xrightarrow{T} & (X_0, \mathcal{B}, \mu) \\ \pi \downarrow & & \downarrow \pi \\ (Y_0, \mathcal{C}, \nu) & \xrightarrow{S} & (Y_0, \mathcal{C}, \nu) \end{array}$$

commutes, then $h_\mu(T) = h_\nu(S)$. This holds, because the bi-measurable measure-preserving map π preserves all the quantities involved in (19)-(21), including the class of partitions for both systems.

A major class of systems where this is very important are the Bernoulli shifts. These are the standard probability space to measure a sequence of i.i.d. events each with outcomes in $\{0, \dots, N-1\}$ with probabilities $p_0, \dots, p_{N-1}$ respectively. That is: $X = \{0, \dots, N-1\}^{\mathbb{N}_0}$ or $\{0, \dots, N-1\}^{\mathbb{Z}}$, σ is the left-shift, and μ the Bernoulli measure that assigns to every cylinder set $[x_m \dots x_n]$ the mass

$$\mu([x_m \dots x_n]) = \prod_{k=m}^{n} \rho(x_k) \qquad \text{where } \rho(x_k) = p_i \text{ if } x_k = i.$$

For such a Bernoulli shift, the entropy is

$$h_\mu(\sigma) = -\sum_i p_i \log p_i, \tag{22}$$

so two Bernoulli shifts (X, p, μ_p) and $(X', p', \mu_{p'})$ can only be isomorphic if $-\sum_i p_i \log p_i = -\sum_i p'_i \log(p'_i)$. The famous theorem of Ornstein showed that entropy is a complete invariant for Bernoulli shifts:

Theorem 15 (Ornstein 1974 [14], cf. page 105 of [16])**.** *Two Bernoulli shifts (X, p, μ_p) and $(X', p', \mu_{p'})$ are isomorphic if and only if $-\sum_i p_i \log p_i = -\sum_i p'_i \log p'_i$.*

Exercise 5. *Conclude that the Bernoulli shift $\mu_{(\frac14, \frac14, \frac14, \frac14)}$ is isomorphic to $\mu_{(\frac18, \frac18, \frac18, \frac18, \frac12)}$, but that no Bernoulli measure on four symbols can be isomorhic to $\mu_{(\frac15, \frac15, \frac15, \frac15, \frac15)}$*

Let us go back to the definition of entropy, and try to answer the outstanding questions.

Definition 12. *We call a real sequence $(a_n)_{n \geqslant 1}$* **subadditive** *if*

$$a_{m+n} \leqslant a_m + a_n \quad \textit{for all } m, n \in \mathbb{N}.$$

Theorem 16. *If $(a_n)_{n\geqslant 1}$ is subadditive, then $\lim_n \frac{a_n}{n} = \inf_{r\geqslant 1} \frac{a_r}{r}$.*

Proof. Every integer n can be written uniquely as $n = i \cdot r + j$ for $0 \leqslant j < r$. Therefore

$$\limsup_{n\to\infty} \frac{a_n}{n} = \limsup_{i\to\infty} \frac{a_{i\cdot r+j}}{i\cdot r+j} \leqslant \limsup_{i\to\infty} \frac{ia_r + a_j}{i\cdot r+j} = \frac{a_r}{r}.$$

This holds for all $r \in \mathbb{N}$, so we obtain

$$\inf_r \frac{a_r}{r} \leqslant \liminf_n \frac{a_n}{n} \leqslant \limsup_n \frac{a_n}{n} \leqslant \inf_r \frac{a_r}{r},$$

as required. □

Definition 13. *Motivated by the conditional measure $\mu(P|Q) = \frac{\mu(P\cap Q)}{\mu(Q)}$, we define* **conditional entropy** *of a measure μ as*

$$H_\mu(\mathcal{P}|\mathcal{Q}) = -\sum_j \mu(Q_j) \sum_i \frac{\mu(P_i \cap Q_j)}{\mu(Q_j)} \log \frac{\mu(P_i \cap Q_j)}{\mu(Q_j)}, \tag{23}$$

where i runs over all elements $P_i \in \mathcal{P}$ and j runs over all elements $Q_j \in \mathcal{Q}$.

Avoiding philosophical discussions how to interpret this notion, we just list the main properties that are needed in this course that rely of condition entropy:

Proposition 10. *Given measures μ, μ_i and two partitions $\mathcal{P}$ and $\mathcal{Q}$, we have*

1. $H_\mu(\mathcal{P} \vee \mathcal{Q}) \leqslant H_\mu(\mathcal{P}) + H_\mu(\mathcal{Q})$;
2. $H_\mu(T, \mathcal{P}) \leqslant H_\mu(T, \mathcal{Q}) + H_\mu(\mathcal{P} \mid \mathcal{Q})$.
3. $\sum_{i=1}^n p_i H_{\mu_i}(\mathcal{P}) \leqslant H_{\sum_{i=1}^n p_i\mu_i}(\mathcal{P})$ *whenever* $\sum_{i=1}^n p_1 = 1$, $p_i \geqslant 0$,

Subadditivity is the key to the convergence in (20). Call $a_n = H_\mu(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P})$. Then

$$\begin{aligned}
a_{m+n} &= H_\mu(\bigvee_{k=0}^{m+n-1} T^{-k}\mathcal{P}) && \text{use Proposition 10, part 1.}\\
&\leqslant H_\mu(\bigvee_{k=0}^{m-1} T^{-k}\mathcal{P}) + H_\mu(\bigvee_{k=m}^{m+n-1} T^{-k}\mathcal{P}) && \text{use } T\text{-invariance of } \mu\\
&= H_\mu(\bigvee_{k=0}^{m-1} T^{-k}\mathcal{P}) + H_\mu(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P})\\
&= a_m + a_n.
\end{aligned}$$

Therefore $H_\mu(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P})$ is subadditive, and the existence of the limit of $\frac{1}{n}H_\mu(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P})$ follows.

Proposition 11. *Entropy has the following properties:*

1. *The identity map has entropy* 0*;*
2. $h_\mu(T^R) = R \cdot h_\mu(T)$ *and for invertible systems* $h_\mu(T^{-R}) = R \cdot h_\mu(T)$*.*

Proof. Statement 1. follows simply because $\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P} = \mathcal{P}$ if T is the identity map, so the cardinality of $\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}$ doesn't increase with n.

For statement 2. set $\mathcal{Q} = \bigvee_{j=0}^{R-1} T^{-j}\mathcal{P}$. Then for $k \geqslant 1$,

$$\begin{aligned} R \cdot H_\mu(T, \mathcal{P}) &= \lim_{n\to\infty} R \cdot \frac{1}{nR} H_\mu(\bigvee_{j=0}^{nR-1} T^{-k}\mathcal{P}) \\ &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{j=0}^{n-1} (T^R)^{-j}\mathcal{Q}) \\ &= H_\mu(T^R, \mathcal{Q}). \end{aligned}$$

Taking the supremum over all $\mathcal{P}$ or $\mathcal{Q}$ has the same effect. □

The next theorem is the key to really computing entropy, as it shows that a single well-chosen partition $\mathcal{P}$ suffices to compute the entropy as $h_\mu(T) = H_\mu(T, \mathcal{P})$.

Theorem 17 (Kolmogorov-Sinaĭ)**.** *Let* $(X, \mathcal{B}, T, \mu)$ *be a measure-preserving dynamical system. If partition* $\mathcal{P}$ *is such that*

$$\begin{cases} \bigvee_{j=0}^{\infty} T^{-k}\mathcal{P} \text{ generates } \mathcal{B} & \text{if } T \text{ is non-invertible,} \\ \bigvee_{j=-\infty}^{\infty} T^{-k}\mathcal{P} \text{ generates } \mathcal{B} & \text{if } T \text{ is invertible,} \end{cases}$$

then $h_\mu(T) = H_\mu(T, \mathcal{P})$*.*

We haven't explained properly what "generates $\mathcal{B}$ means, but the idea you should have in mind is that (up to measure 0), every two points in X should be in different elements of $\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}$ (if T is non-invertible), or of $\bigvee_{k=-n}^{n-1} T^{-k}\mathcal{P}$ (if T is invertible) for some sufficiently large n. The partition $\mathcal{B} = \{X\}$ fails miserably here, because $\bigvee_{j=-n}^{n} T^{-k}\mathcal{P} = \mathcal{P}$ for all n and no two points are ever separated in $\mathcal{P}$. A more subtle example can be created for the doubling map $T_2 : \mathbb{S}^1 \to \mathbb{S}^1$, $T_2(x) = 2x \pmod 1$. The partition $\mathcal{P} = \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$. is separating every two points, because if $x \neq y$, say $2^{-(n+1)} < |x - y| \leqslant 2^{-n}$, then there is $k \leqslant n$ such that $T_2^k x$ and $T_2^k y$ belong to different partition elements.

On the other hand, $\mathcal{Q} = \{[\frac{1}{4}, \frac{3}{4}), [0, \frac{1}{4}) \cup [\frac{3}{4}, 1)\}$ does **not** separate points. Indeed, if $y = 1 - x$, then $T_2^k(y) = 1 - T_2^k(x)$ for all $k \geqslant 0$, so x and y belong to the same partition element, $T_2^k(y)$ and $T_2^k(x)$ will also belong to the same partition element!

In this case, $\mathcal{P}$ can be used to compute $h_\mu(T)$, while $\mathcal{Q}$ in principle cannot (although here, for all Bernoulli measure $\mu = \mu_{p,1-p}$, we have $h_\mu(T_2) = H_\mu(T, \mathcal{P}) = H_\mu(T, \mathcal{Q})$).

Proof of Theorem 17. Let $\mathcal{A}$ be the generating partition. Then $h_\mu(T, \mathcal{A}) \leqslant h_\mu(T)$ because the RHS is the supremum over all partitions. Let $\varepsilon > 0$ be arbitrary, and let $\mathcal{C}$ be a finite partition, say $\#\mathcal{C} = N$, such that $H_\mu(T, \mathcal{C}) \geqslant h_\mu(T) - \varepsilon$. We have

$$\begin{aligned} h_\mu(T| \bigvee_{i=-k}^{k} T^i \mathcal{A}) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{j=0}^{n-1} T^{-j}(\bigvee_{i=-k}^{k} T^i\mathcal{A})) = \lim_{n\to\infty} \frac{1}{n} H_\mu(T^k(\bigvee_{j=0}^{n+2k} T^{-j}\mathcal{A})) \\ &= \lim_{n\to\infty} \frac{n+2k+1}{n} \frac{1}{n+2k+1} H_\mu(\bigvee_{j=0}^{n+2k} T^{-j}\mathcal{A}) = h_\mu(T\mathcal{A}). \end{aligned}$$

Using this and Proposition 10, part 2, we compute:

$$h_\mu(T|\mathcal{C}) \leqslant h_\mu(T| \bigvee_{i=-k}^{k} T^i\mathcal{A}) + H_\mu(\mathcal{C}| \bigvee_{i=-k}^{k} T^i\mathcal{A}) = h_\mu(T|\mathcal{A}) + H_\mu(\mathcal{C}| \bigvee_{i=-k}^{k} T^i\mathcal{A}).$$

Since $\mathcal{A}$ is generating, the measure of points in X for which the element $A \in \bigvee_{i=-k}^{k} T^i k\mathcal{A}$ that contains x is itself contained in a single element of $\mathcal{C}$ tends to one as $k \to \infty$. Therefore we can find k so large that if we set

$$\mathcal{A}^* = \{A \in \bigvee_{i=-k}^{k} T^i k\mathcal{A} : A \cap C \neq \varnothing \neq A \cap C' \text{ for some } C \neq C' \in \mathcal{C}\},$$

then $\mu(\bigcup_{A\in\mathcal{A}^*} A) \leqslant \varepsilon/(N \log N)$. This gives

$$\begin{aligned} H_\mu(\mathcal{C}| \bigvee_{i=-k}^{k} T^i k\mathcal{A}) &= \sum_{A\in\bigvee_{i=-k}^{k} T^i k\mathcal{A}} \sum_{C\in\mathcal{C}} \mu(A)\varphi\left(\frac{\mu(A\cap C)}{\mu(A)}\right) \\ &= \sum_{A\in\mathcal{A}^*} \mu(A) \sum_{C\in\mathcal{C}} \varphi\left(\frac{\mu(A\cap C)}{\mu(A)}\right) \leqslant \sum_{A\in\mathcal{A}^*} \mu(A) N \log N < \varepsilon. \end{aligned}$$

Combining all, we get $h_\mu(T, \mathcal{A}) \geqslant h_\mu(T, \mathcal{C}) - \varepsilon \geqslant h_\mu(T) - 2\varepsilon$ completing the proof. $\square$

We finish this section with computing the entropy for a Bernoulli shift on two symbols, *i.e.,* we will prove (22) for two-letter alphabets and any probability $\mu([0]) =: p \in [0, 1]$. The space is thus $X = \{0, 1\}^{\mathbb{N}_0}$ and each $x \in X$ represents an infinite sequence of coin-flips with an unfair coin that gives head probability p (if head has the symbol 0). Recall from probability theory

$$\mathbb{P}(k \text{ heads in } n \text{ flips}) = \binom{n}{k} p^k (1-p)^{n-k},$$

so by full probability:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1.$$

Here $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are the binomial coefficients, and we can compute

$$\begin{cases} k\binom{n}{k} = \frac{n!}{(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1} \\ (n-k)\binom{n}{k} = \frac{n!}{(k)!(n-k-1)!} = n\frac{(n-1)!}{k!(n-k-1)!} = n\binom{n-1}{k} \end{cases} \tag{24}$$

This gives all the ingredients necessary for the computation.

$$\begin{aligned} H_\mu(\bigvee_{k=0}^{n-1} \sigma^{-k}\mathcal{P}) &= -\sum_{x_0,\dots,x_{n-1}=0}^{1} \mu([x_0,\dots,x_{n-1}]) \log \mu([x_0,\dots,x_{n-1}]) \\ &= -\sum_{x_0,\dots,x_{n-1}=0}^{1} \prod_{j=0}^{n-1} \rho(x_j) \log \prod_{j=0}^{n-1} \rho(x_j) \\ &= -\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \log \left(p^k (1-p)^{n-k}\right) \\ &= -\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} k \log p \\ &\qquad -\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (n-k) \log(1-p) \end{aligned}$$

In the first sum, the term $k=0$ gives zero, as does the term $k=n$ for the second sum. Thus we leave out these terms and rearrange by (24):

$$\begin{aligned} &= -p \log p \sum_{k=1}^{n} k \binom{n-1}{k} p^{k-1} (1-p)^{n-k} \\ &\qquad -(1-p)\log(1-p) \sum_{k=0}^{n-1} (n-k) \binom{n}{k} p^k (1-p)^{n-k-1} \\ &= -p \log p \sum_{k=1}^{n} n \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \\ &\qquad -(1-p)\log(1-p) \sum_{k=0}^{n-1} n \binom{n-1}{k} p^k (1-p)^{n-k-1} \\ &= n\left(-p\log p - (1-p)\log(1-p)\right). \end{aligned}$$

The partition $\mathcal{P} = \{[0],[1]\}$ is generating, so by Theorem 17,

$$h_\mu(\sigma) = H_\mu(\sigma, \mathcal{P}) = \lim_n \frac{1}{n} H_\mu(\bigvee_{k=0}^{n-1} \sigma^{-k}\mathcal{P}) = -p\log p - (1-p)\log(1-p)$$

as required.

15 The Variational Principle

The Variational Principle claims that topological entropy (or pressure) is achieved by taking the supremum of the measure-theoretic entropies over all invariant probability measures. But in the course of these notes, topological entropy has seen various definitions. Even $\sup\{h_\mu(T) : \mu$ is a T-invariant probability measure$\}$ is sometimes used as definition of topological entropy. So it is time to be more definite.

We will do this by immediately passing to topological pressure, which we will base on the definition in terms of (n, δ)-spanning sets and/or (n, ε)-separated sets. Topological entropy then simply emerges as $h_{top}(T) = P_{top}(T, 0)$.

Theorem 18 (The Variational Principle)**.** *Let (X, d) be a compact metric space, $T : X \to X$ a continuous map and $\psi : X \to \mathbb{R}$ as continuous potential. Then*

$$P_{top}(T, \psi) = \sup\{h_\mu(T) + \int_X \psi \, d\mu \ : \ \mu \text{ is a } T\text{-invariant probability measure}\}. \tag{25}$$

Remark 3. *By the ergodic decomposition, every T-invariant probability measure can be written as convex combination (sometimes in the form of an integral) of ergodic T-invariant probability measures. Therefore, it suffices to take the supremum over all ergodic T-invariant probability measures.*

Proof. First we show that for every T-invariant probability measure, $h_\mu(T) + \int_X \psi \, d\mu \leqslant P_{top}(T, \psi)$. Let $\mathcal{P} = \{P_0, \dots, P_{N-1}\}$ be an arbitrary partition with $N \geqslant 2$ (if $\mathcal{P} = \{X\}$, then $h_\mu(T, \mathcal{P}) = 0$ and there is not much to prove). Let $\eta > 0$ be arbitrary, and choose $\varepsilon > 0$ so that $\varepsilon N \log N < \eta$.

By "regularity of μ", there are compact sets $Q_i \subset P_i$ such that $\mu(P_i \setminus Q_i) < \varepsilon$ for each $0 \leqslant i < N$. Take $Q_N = X \setminus \cup_{i=0}^{N-1} Q_i$. Then $\mathcal{Q} = \{Q_0, \dots, Q_N\}$ is a new partition of X, with $\mu(Q_N) \leqslant N\varepsilon$. Furthermore

$$\frac{\mu(P_i \cap Q_j)}{\mu(Q_j)} = \begin{cases} 0 & \text{if } i \neq j < N, \\ 1 & \text{if } i = j < N. \end{cases}$$

whereas $\sum_{i=0}^{N-1} \frac{\mu(P_i \cap Q_N)}{\mu(Q_N)} = 1$. Therefore the conditional entropy

$$\begin{aligned} H_\mu(\mathcal{P}|\mathcal{Q}) &= \sum_{j=0}^{N} \sum_{i=0}^{N-1} \mu(Q_j) \underbrace{\varphi\left(\frac{\mu(P_i \cap Q_j)}{\mu(Q_j)}\right)}_{= \ 0 \text{ if } j<N} \\ &= -\mu(Q_N) \sum_{i=0}^{N-1} \frac{\mu(P_i \cap Q_N)}{\mu(Q_N)} \log(\frac{\mu(P_i \cap Q_N)}{\mu(Q_N)}) \\ &\leqslant \mu(Q_N) \log N \qquad \text{by Corollary 3} \\ &\leqslant \varepsilon N \log N < \eta. \end{aligned}$$

Choose $0 < \delta < \frac{1}{2}\min_{0 \leqslant i < j < N} d(Q_i, Q_j)$ so that

$$d(x,y) < \delta \text{ implies } |\psi(x) - \psi(y)| < \varepsilon. \tag{26}$$

Here we use uniform continuity of ψ on the compact space X. Fix n and let $E_n(\delta)$ be an (n,δ)-spanning set. For $Z \in \mathcal{Q}_n := \bigvee_{k=0}^{n-1} T^{-k}\mathcal{Q}$, let $\alpha(Z) = \sup\{S_n\psi(x) \ : \ x \in Z\}$. For each such Z, also choose $x_Z \in \overline{Z}$ such that $S_n\psi(x) = \alpha(Z)$ (again we use continuity of ψ here), and $y_Z \in E_n(\delta)$ such that $d_n(x_Z, y_Z) < \delta$. Hence

$$\alpha(Z) - n\varepsilon \leqslant S_n\psi(y_Z) \leqslant \alpha(Z) + n\varepsilon.$$

This gives

$$H_\mu(\mathcal{Q}_n) + \int_X S_n\psi \ d\mu \leqslant \sum_{Z \in \mathcal{Q}_n} \mu(Z)(\alpha(Z) - \log\mu(Z)) \leqslant \log \sum_{Z \in \mathcal{Q}_n} e^{\alpha(Z)} \tag{27}$$

by Corollary 4.

Each δ-ball intersects the closure of at most two elements of $\mathcal{Q}$. Hence, for each $y \in E_n(\delta)$, the cardinality $\#\{Z \in \mathcal{Q}_n : y_Z = y\} \leqslant 2^n$. Therefore

$$\sum_{Z \in \mathcal{Q}_n} e^{\alpha(Z) - n\varepsilon} \leqslant \sum_{Z \in \mathcal{Q}_n} e^{S_n\psi(y_Z)} \leqslant 2^n \sum_{y \in E_n(\delta)} e^{S_n\psi(y)}.$$

Take the logarithm and rearrange to

$$\log \sum_{Z \in \mathcal{Q}_n} e^{\alpha(Z)} \leqslant n(\varepsilon + \log 2) + \log \sum_{y \in E_n(\delta)} e^{S_n\varphi(y)}.$$

By T-invariance of μ we have $\int S_n\psi \ d\mu = n\int \psi \ d\mu$. Therefore

$$\begin{aligned} \frac{1}{n}H_\mu(\mathcal{Q}_n) + \int_X \psi \ d\mu &\leqslant \frac{1}{n}H_\mu(\mathcal{Q}_n) + \frac{1}{n}\int_X S_n\psi \ d\mu \\ &\leqslant \frac{1}{n}\log \sum_{Z \in \mathcal{Q}_n} e^{\alpha(Z)} \\ &\leqslant \varepsilon + \log 2 + \frac{1}{n}\log \sum_{y \in E_n(\delta)} e^{S_n\varphi(y)}. \end{aligned}$$

Taking the limit $n \to \infty$ and recalling that $E_n(\delta)$ is an arbitrary (n,δ)-spanning set, gives

$$H_\mu(T, \mathcal{Q}) + \int_X \psi \ d\mu \leqslant \varepsilon + \log 2 + P_{top}(T, \psi).$$

By Proposition 10, part 2., and recalling that $\varepsilon < \eta$, we get

$$H_\mu(T, \mathcal{P}) + \int_X \psi \ d\mu = H_\mu(T, \mathcal{Q}) + H_\mu(\mathcal{P}|\mathcal{Q}) + \int_X \psi \ d\mu \leqslant 2\eta + \log 2 + P_{top}(T, \psi).$$

We can apply the same reasoning to T^R and $S_R\psi$ instead of T and ψ. This gives

$$\begin{aligned} R\cdot\left(H_\mu(T,\mathcal{P})+\int_X \psi\ d\mu\right) &= H_\mu(T^R,\mathcal{P})+\int_X S_R\psi\ d\mu \\ &\leqslant 2\eta+\log 2+P_{top}(T^R,S_R\psi) \\ &= 2\eta+\log 2+R\cdot P_{top}(T,\psi). \end{aligned}$$

Divide by R and take $R\to\infty$ to find $H_\mu(T,\mathcal{P})+\int_X \psi\ d\mu \leqslant P_{top}(T,\psi)$. Finally take the supremum over all partitions $\mathcal{P}$.

Now the other direction, we will work with (n,ε)-separated sets. After choosing $\varepsilon>0$ arbitrary, we need to find a T-invariant probability measure μ such that

$$h_\mu(T)+\int_X \psi\ d\mu \geqslant \limsup_{n\to\infty}\frac{1}{n}\log K_n(T,\psi,\varepsilon) := P(T,\psi,\varepsilon).$$

Let $E_n(\varepsilon)$ be an (n,ε)-separated set such that

$$\log\sum_{y\in E_n(\varepsilon)} e^{S_n\psi(y)} \geqslant \log K_n(T,\psi,\varepsilon)-1. \tag{28}$$

Define Δ_n as weighted sum of Dirac measures:

$$\Delta_n=\frac{1}{\mathcal{Z}}\sum_{y\in E_n(\varepsilon)} e^{S_n\psi(y)}\delta_y,$$

where $\mathcal{Z}=\sum_{y\in E_n(\varepsilon)} e^{S_n\psi(y)}$ is the normalising constant. Take a new probability measure

$$\mu_n=\frac{1}{n}\sum_{k=0}^{n-1}\Delta_n\circ T^{-k}.$$

Therefore

$$\begin{aligned} \int_X \psi\ d\mu_n &= \frac{1}{n}\sum_{k=0}^{n-1}\int_X \psi\ d(\Delta_n\circ T^{-k}) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{y\in E_n(\varepsilon)}\psi\circ T^k(y)\frac{1}{\mathcal{Z}}e^{S_n\psi(y)} \\ &= \frac{1}{n}\sum_{y\in E_n(\varepsilon)} S_n\psi(y)\frac{1}{\mathcal{Z}}e^{S_n\psi(y)} = \frac{1}{n}\int_X S_n\psi\ d\Delta_n. \end{aligned} \tag{29}$$

Since the space of probability measures on X is compact in the weak topology, we can find a sequence $(n_j)_{j\geqslant 1}$ such that for every continuous function $f:X\to\mathbb{R}$

$$\int_X f\ d\mu_{n_j}\to\int_X f\ d\mu \quad \text{as } j\to\infty.$$

Choose a partition $\mathcal{P} = \{P_0, \ldots, P_{N-1}\}$ with $\mathrm{diam}(P_i) < \varepsilon$ and $\mu(\partial P_i) = 0$ for all $0 \leqslant i < N$. Since $Z \in \mathcal{P}_n := \bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}$ contains at most one element of an (n, ε)-separated set, we have

$$\begin{aligned} H_{\Delta_n}(\mathcal{P}_n) + \int_X S_n\psi \; d\Delta_n &= \sum_{y \in E_n(\varepsilon)} \Delta_n(\{y\}) \left(S_n\psi(y) - \log \Delta_n(\{y\})\right) \\ &= \log \sum_{y \in E_n(\varepsilon)} e^{S_n\psi(y)} = \log \mathcal{Z}. \end{aligned}$$

by Corollary 4

Take $0 < q < n$ arbitrary, and for $0 \leqslant j < q$, let

$$U_j = \{j, j+1, \ldots, a_j q + j - 1\} \qquad \text{where } a_j = \lfloor \frac{n-j}{q} \rfloor.$$

Then

$$\{0, 1, \ldots, n-1\} = U_j \cup \underbrace{\{0, 1, \ldots, j-1\} \cup a_j q + j, a_j q + j + 1, \ldots, n-1\}}_{V_j}$$

where V_j has at most $2q$ elements. We split

$$\begin{aligned} \bigvee_{k=0}^{n-1} T^{-k}\mathcal{P} &= \left(\bigvee_{r=0}^{a_j-1} \bigvee_{i=0}^{q-1} T^{-(rq+j+i)}\mathcal{P} \right) \vee \bigvee_{l \in V_j} T^{-l}\mathcal{P} \\ &= \left(\bigvee_{r=0}^{a_j-1} T^{-(rq+j)} \bigvee_{i=0}^{q-1} T^{-i}\mathcal{P} \right) \vee \bigvee_{l \in V_j} T^{-l}\mathcal{P}. \end{aligned}$$

Therefore,

$$\begin{aligned} \log \mathcal{Z} &= H_{\Delta_n}(\mathcal{P}_n) + \int_X S_n\psi \; d\Delta_n \\ &\leqslant \sum_{r=0}^{a_j-1} H_{\Delta_n}(T^{-(rq+j)} \bigvee_{i=0}^{q-1} T^{-i}\mathcal{P}) + H_{\Delta_n}(\bigvee_{l \in V_j} T^{-l}\mathcal{P}) + \int_X S_n\psi \; d\Delta_n \\ &\leqslant \sum_{r=0}^{a_j-1} H_{\Delta_n \circ T^{-(rq+j)}}(\bigvee_{i=0}^{q-1} T^{-i}\mathcal{P}) + 2q \log N + \int_X S_n\psi \; d\Delta_n, \end{aligned}$$

because $\bigvee_{l \in V_j} T^{-l}\mathcal{P}$ has at most N^{2q} elements and using Corollary 3. Summing the above inequality over $j = 0, \ldots, q-1$, gives

$$\begin{aligned} q \log \mathcal{Z} &= \sum_{j=0}^{q-1} \sum_{r=0}^{a_j-1} H_{\Delta_n \circ T^{-rq+j}}(\bigvee_{i=0}^{q-1} T^{-i}\mathcal{P}) + 2q^2 \log N + q \int_X S_n\psi \; d\Delta_n \\ &\leqslant n \sum_{k=0}^{n-1} \frac{1}{n} \; H_{\Delta_n \circ T^{-k}}(\bigvee_{i=0}^{q-1} T^{-i}\mathcal{P}) + 2q^2 \log N + q \int_X S_n\psi \; d\Delta_n. \end{aligned}$$

Proposition 10, part 3., allows us to swap the weighted average and the operation H:

$$q \log \mathcal{Z} \leqslant n H_{\mu_n}(\bigvee_{i=0}^{q-1} T^{-i}\mathcal{P}) + 2q^2 \log N + q \int_X S_n\psi \, d\Delta_n.$$

Dividing by n and recalling (28) for the left hand side, and (29) to replace Δ_n by μ_n, we find

$$\frac{q}{n} \log K_n(T, \psi, \varepsilon) - \frac{q}{n} \leqslant H_{\mu_n}(\bigvee_{i=0}^{q-1} T^{-i}\mathcal{P}) + \frac{2q^2}{n} \log N + q \int_X \psi \, d\mu_n.$$

Because $\mu(\partial P_i) = 0$ for all i, we can replace n by n_j and take the weak limit as $j \to \infty$. This gives

$$qP(T, \psi, \varepsilon) \leqslant H_\mu(\bigvee_{i=0}^{q-1} T^{-i}\mathcal{P}) + q \int_X \psi \, d\mu.$$

Finally divide by q and let $q \to \infty$:

$$P(T, \psi, \varepsilon) \leqslant h_\mu(T) + \int_X \psi \, d\mu.$$

This concludes the proof. □

16 Measures of maximal entropy

16.1 Subshifts of finite type

To each directed graph $(G, \to)$, say with vertices $\{1, \dots, N\}$ one can assign a *transition matrix* $A = (a_{i,j})_{i,j=1}^N$ where for each i, j, $A_{i,j}$ counts the number of edges from vertex i to vertex j. We call G *irreducible* if there exists a path (of some length) from each vertex to each vertex. It is called *aperiodic* if for each i, j there is $m \in \mathbb{N}$ such that there is a path from i to j of length n for every $n \geqslant m$. In terms of the transmatrix, this translates to: A is irreducible if for every i, j there is n such that $A^n_{i,j} > 0$, and A is aperiodic if in addition there is n such that $A^n_{i,j} > 0$ for all i, j.

The set of (bi)inifnite strings

$$\Sigma_A = \{(x_i)_{i \in \mathbb{Z}} : x_i \in \{1, \dots, N\}, A_{x_i, x_{i+1}} > 0 \text{ for all } i \in \mathbb{Z}\}$$

is shift-invariant and closed in the standard product topology of $\{1, \dots, N\}^{\mathbb{Z}}$. Hence it is a *subshift*. It is called *subshift of finite type (SFT)* because of the finite collection of fobidden words (namely the pairs ij such that $A_{i,j} = 0$) that fully determines Σ_A.

It is easy to see that the *word-complexity*

$$\begin{aligned} p_n(\Sigma_A) &:= \#\{x_0 \dots x_{n-1} \text{ subword appearing in } \Sigma_A\} \\ &= \#\{\text{paths of length } n-1 \text{ in } G\} = \sum_{i,j} A^n_{i,j}. \end{aligned}$$

Because the partition into n-cylinders forms an open 2^{-n}-cover of Σ_A, we can derive

$$h_{top}(\sigma|_{\Sigma_A}) = \lim_{n\to\infty} \frac{1}{n} \log p_n(\Sigma_A) = \log \lambda,$$

where λ is the leading eigenvalue of the transition matrix A. That A has a unique, positive, leading eigenvalue follows from the following theorem.

Theorem 19 (Perron-Frobenius)**.** *Let A be a nonnegative $N \times N$-matrix such that $A^n > 0$ for some $m \in \mathbb{N}$. Then A has a unique (up to scaling) eigenvector with all entries > 0. The corresponding eigenvalue is positive, has multiplicity one, and is larger than the absolute value of all other eigenvakues of A.*

Proof. Let $C = \mathbb{R}^N_{\geqslant 0}$ be the one-sided cone of nonegative vectors. Since A is nonegative, $AC \subset C$, and because $A^m > 0$, $A^m C \subset A^o$, by which we mean that every nonzero vector in C is mapped into the interior of C by A^m. Define on the simplex $S = \{x \in C : \|x\| = 1\}$ the map $f : S \to S$ by $f(x) = Ax/\|Ax\|$. Since $A^n > 0$, it is impossible that $Ax = 0$ for $x \in S$, so f is well-defined. Although nonlinear, the map f is convex, meaning that it sends convex subsets of S to convex subsets, and extremal points to extremal points. Applying this to $\Pi_n := \cap_{k=0}^n f^k(S)$, we conclude that (Π_n) is a nested sequence of convex sets with $f^n(e_i)$, $i = 1, \dots N$ as extremal points. This carries over to the limit $\Pi := \bigcap_n \Pi_n$ as well; note that Π is contained in the interior of S because $A^n > 0$. We can select a subsequence (n_j) such that $f^{n_j}(e_i) \to p_i$ are the extremal points of Π. This is a finite set, invariant under f, so there is M such that each p_i is fixed by f^M and therefore an eigenvector of A^M associatedto a poistive eigenvalue. By reordering the p_i, we can assume that the corresponding eigenvalues of A^M are $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_N$.

1. If $\lambda_2 = \lambda_1$ and $p_1 \neq p_2$, then we can find $v = \alpha_1 p_1 + \alpha_2 p_2 \in \partial C$. This is also an eigenvector of A^M, so $A^{kM} v \in \partial C$ for all k, but this contradicts that $A^m C \in C^o$.

2. If $\lambda_2 < \lambda_1$, then take $v = p_2 - \varepsilon p_1 \in C$ (for $\varepsilon > 0$ sufficiently small), and note that $A^{kM} v = \lambda_2^{kM} p_2 - \varepsilon \lambda_1^{kM} p_1$ cannot be contained in C for all k. This contradicts again the invariance of C. Hence, all p_i coincide, and it is the unique fixed point of f.

3. To show that λ_1 has multiplicity one, assume by contradiction that there is a generalised eigenvector $v \in S$ with $A^M v = \lambda v + p_1$. Then also $A^{kM} v = \lambda^k v + k\lambda^{k-1} p_1$. Take $w = p_1 - \varepsilon v \in C$ for some small $\varepsilon > 0$. Then $A^{kM} w = \lambda^{k-1}(\lambda - \varepsilon k) v - \varepsilon \lambda^n v$ which cannot be contained in c for large k. This again contradticts that $A^M C \subset C$.

4. Finally, suppose that μ is some eigenvalue, not necessarily associated with an eigenvector in S, such that $|\mu| > \lambda_1$. There is a A^M-invariant subspace V (possibly of dimension two if $\mu \notin \mathbb{R}$) such that $A^M : V \to V$ is the composition of an isometry and a dilatation by a factor $|\mu|$. In particular, there is a subsequence (k_j) such that $|\mu|^{-k_j} A^{k_j M} v \to v$ for every $v \in V$. Take $v \in V$ so that $w := v + p_1 \in \partial C$. Then $|\mu|^{-k_j} A^{k_j M} w \to w$, contradicting that $A^m C \subset C^o$. Hence all other eigenvectors of A^M are strictly smaller than λ_1.

The proof now follows by taking $\lambda = \lambda_1^{1/M}$. □

16.2 Parry measure

For the full shift (Σ, σ) with $\Sigma = \{0, \dots, N-1\}^{\mathbb{N}_0}$ or $\Sigma = \{0, \dots, N-1\}^{\mathbb{Z}}$, we have $h_{top}(\sigma) = \log N$, and the $(\frac{1}{N}, \dots, \frac{1}{N})$-Bernoulli measure μ indeed achieves this maximum: $h_\mu(\sigma) = h_{top}(\sigma)$. Hence μ is a (and in this case unique) **measure of maximal entropy**. The intuition to have here is that for a measure to achieve maximal entropy, it should distribute its mass as evenly over the space as possible. But how does this work for subshifts, where it is not immediately obvious how to distribute mass evenly?

For subshifts of finite type, Parry [15] demonstrated how to construct the measure of maximal entropy, which is now called after him. Let (Σ_A, σ) be a subshift of finite type on alphabet $\{0, \dots, N-1\}$ with transition matrix $A = (A_{i,j})_{i,j=0}^{N-1}$, so $x = (x_n) \in \Sigma_n$ if and only if $A_{x_n, x_{n+1}} = 1$ for all n. Let us assume that A is aperiodic and irreducible. Then by the Perron-Frobenius Theorem for matrices, there is a unique real eigenvalue, of multiplicity one, which is larger in absolute value than every other eigenvalue, and $h_{top}(\sigma) = \log \lambda$. Furthermore, by irreducibility of A, the left and right eigenvectors $u = (u_0, \dots, u_{N-1})$ and $v = (v_0, \dots, v_{N-1})^T$ associated to λ are unique up to a multiplicative factor, and they can be chosen to be strictly positive. We will scale them such that

$$\sum_{i=0}^{N-1} u_i v_i = 1.$$

Now define the **Parry measure** by

$$\begin{aligned} p_i &:= u_i v_i = \mu([i]), \\ p_{i,j} &:= \frac{A_{i,j} v_j}{\lambda v_i} = \mu([ij] \mid [i]), \end{aligned}$$

so $p_{i,j}$ indicates the conditional probability that $x_{n+1} = j$ knowing that $x_n = i$. Therefore $\mu([ij]) = \mu([i])\mu([ij] \mid [i]) = p_i p_{i,j}$. It is stationary (*i.e.*, shift-invariant) but not quite a product measure, but $\mu([i_m \dots i_n]) = p_{i_m} \cdot p_{i_m, i_{m+1}} \cdots p_{i_{n-1}, i_n}$.

Theorem 20. *The Parry measure μ is the unique measure of maximal entropy for a subshift of finite type with aperiodic irreducible transition matrix.*

Proof. In this proof, we will only show that $h_\mu(\sigma) = h_{top}(\sigma) = \log \lambda$, and skip the (more complicated) uniqueness part.

The definitions of mass of 1-cylinders and 2-cylinders are compatible, because (since v is a right eigenvector)

$$\sum_{j=0}^{N-1} \mu([ij]) = \sum_{j=0}^{N-1} p_i p_{i,j} = p_i \sum_{j=0}^{N-1} \frac{A_{i,j} v_j}{\lambda v_i} = p_i \frac{\lambda v_i}{\lambda v_i} = p_i = \mu([i]).$$

Summing over i, we get $\sum_{i=0}^{N-1} \mu([i]) = \sum_{i=0}^{N-1} u_i v_i = 1$, due to our scaling.

To show that μ is shift-invariant, we take any cylinder set $Z = [i_m \dots i_n]$ and compute

$$\begin{aligned}
\mu(\sigma^{-1} Z) &= \sum_{i=0}^{N-1} \mu([i i_m \dots i_n]) = \sum_{i=0}^{N-1} \frac{p_i p_{i,i_m}}{p_{i_m}} \mu([i_m \dots i_n]) \\
&= \mu([i_m \dots i_n]) \sum_{i=0}^{N-1} \frac{u_i v_i \; A_{i,i_m} v_{i_m}}{\lambda v_i \; u_{i_m} v_{i_m}} \\
&= \mu(Z) \sum_{i=0}^{N-1} \frac{u_i A_{i,i_m}}{\lambda u_{i_m}} = \mu(Z) \frac{\lambda u_{i_m}}{\lambda u_{i_m}} = \mu(Z).
\end{aligned}$$

This invariance carries over to all sets in the σ-algebra $\mathcal{B}$ generated by the cylinder sets.

Based on the interpretation of conditional probabilities, the identity

$$\sum_{\substack{i_{m+1},\dots,i_n=0 \\ A_{i_k,i_{k+1}}=1}}^{N-1} p_{i_m} p_{i_m,i_{m+1}} \cdots p_{i_{n-1},i_n} = p_{i_m} \text{ and } \sum_{\substack{i_m,\dots,i_{n-1}=0 \\ A_{i_k,i_{k+1}}=1}}^{N-1} p_{i_m} p_{i_m,i_{m+1}} \cdots p_{i_{n-1},i_n} = p_{i_n} \qquad (30)$$

follows because the left hand side indicates the total probability of starting in state i_m and reach some state after $n-m$ steps, respectively start at some state and reach state n after $n-m$ steps.

To compute $h_\mu(\sigma)$, we will confine ourselves to the partition $\mathcal{P}$ of 1-cylinder sets; this partition is generating, so this restriction is justified by Theorem 17.

$$\begin{aligned}
H_\mu(\bigvee_{k=0}^{n-1} \sigma^{-k} \mathcal{P}) &= -\sum_{\substack{i_0,\dots,i_{n-1}=0 \\ A_{i_k,i_{k+1}}=1}}^{N-1} \mu([i_0 \dots i_{n-1}]) \log \mu([i_0 \dots i_{n-1}]) \\
&= -\sum_{\substack{i_0,\dots,i_{n-1}=0 \\ A_{i_k,i_{k+1}}=1}}^{N-1} p_{i_0} p_{i_0,i_1} \cdots p_{i_{n-1},i_n} \left(\log p_{i_0} + \log p_{i_0,i_1} + \dots + \log p_{i_{n-2},i_{n-1}} \right) \\
&= -\sum_{i_0=0}^{N-1} p_{i_0} \log p_{i_0} - (n-1) \sum_{i,j=0}^{N-1} p_i p_{i,j} \log p_{i,j},
\end{aligned}$$

by (30) used repeatedly. Hence

$$\begin{aligned} h_\mu(\sigma) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{k=0}^{n-1} \sigma^{-k}\mathcal{P}) \\ &= -\sum_{i,j=0}^{N-1} p_i p_{i,j} \log p_{i,j} \\ &= -\sum_{i,j=0}^{N-1} \frac{u_i A_{i,j} v_j}{\lambda} \left(\log A_{i,j} + \log v_j - \log v_i - \log \lambda\right). \end{aligned}$$

The first term in the brackets is zero because $A_{i,j} \in \{0,1\}$. The second term (summing first over i) simplifies to

$$-\sum_{j=0}^{N-1} \frac{\lambda u_j v_j}{\lambda} \log v_j = -\sum_{j=0}^{N-1} u_j v_j \log v_j,$$

whereas the third term (summing first over j) simplifies to

$$\sum_{i=0}^{N-1} \frac{u_i \lambda v_i}{\lambda} \log v_i = \sum_{i=0}^{N-1} u_i v_i \log v_i.$$

Hence these two terms cancel each other. The remaining term is

$$\sum_{i,j=0}^{N-1} \frac{u_i A_{i,j} v_j}{\lambda} \log \lambda = \sum_{i=0}^{N-1} \frac{u_i \lambda v_i}{\lambda} \log \lambda = \sum_{i=0}^{N-1} u_i v_i \log \lambda = \log \lambda.$$

□

Remark 4. *There are systems without maximising measure, for example among the "shifts of finite type" on infinite alphabets. To give an example (without proof, see [7]), if $\mathbb{N}$ is the alphabet, and the infinite transition matrix $A = (A_{i,j})_{i,j\in\mathbb{N}}$ is given by*

$$A_{i,j} = \begin{cases} 1 & \text{if } j \geqslant i-1, \\ 0 & \text{if } j < i-1, \end{cases}$$

then $h_{top}(\sigma) = \log 4$, but there is no measure of maximal entropy.

Exercise 6. *Find the maximal measure for the Fibonacci subshift of finite type. What is the limit frequency of the symbol zero in μ-typical sequences x?*

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