Exercises for Ergodic Theory 1, WS2020

For the next four exercises, define $T : [0,1] \to [0,1]$ as $T(x) = 10x \bmod 1$.

Exercise 1. Prove that $T^{1000} \neq \frac{1}{2}$ for $x = \sqrt{2} - 1$.

Exercise 2. Prove: $x \in [0,1] \cap \mathbb{Q}$ if and only if x is (eventually) periodic.

Exercise 3. Show that Champernowne's number $x = 0.123456789010111213\dots$ has a recurrent orbit: $x \in \overline{\text{orb}(T(x))}$.

Exercise 4. Show that a normal number in base 10 has a dense orbit under $T(x) = 10x \bmod 1$.

Exercise 5. Every rational number $x \in [0,1] \cap \mathbb{Q}$ is not normal w.r.t. any base.

Exercise 6. Complete the proof of Borel's Theorem that Lebesgue-a.e. $x \in [0,]$ is normal, using Birkhoff's Ergodic Theorem.

Exercise 7. Show that (i) continuity and (ii) compactness of the space are essential assumptiona in the Theorem of Krylov-Bogol'ubov.

Exercise 8. Let $f : X \to X$ and $g : Y \to Y$ be two conjugate dynamical systems. Show that the following properties are preserved by the conjugacy φ.

- $x \in X$ is periodic;
- $x \in X$ is strictly preperiodic;
- $x \in X$ has a dense orbit;
- In case f and g are defined on $[0,1]$ and both are differentiable: $x \in X$ is a neutral periodic point (i.e., $|Df^p(x)| = 1$ for the period p of x).

Exercise 9. Show that Lebesgue measure is not ergodic for the circle rotation R_α if $\alpha \in \mathbb{Q}$.

Exercise 10. Let $\Omega(f)$ be the non-wandering set of a continuous map $f : X \to X$. Show that $\Omega(f)$ is closed, $f(\Omega_f) \subset \Omega_f$ but $f^{-1}(\Omega_f) \subset \Omega_f$ fails in general.

Exercise 11. Consider the first digits of the elements of the sequence $a_n = 2^n$. Does 9 ever appear as first digit? Does digit 2 appear infinitely often? With which frequency does digit 1 appear?

Exercise 12. Show that the unique invariant measure of a uniquely ergodic system is necessarily ergodic.

Exercise 13. Show that the condition of ergodicity of invariant probability measures is essential for the following Proposition "$\mu \ll \nu$ and ν ergodic $\to \mu = \nu$". If $\mu \ll \nu$ and μ is ergodic. Does it follow that $\mu = \nu$?

Exercise 14. Compute the average frequency of the digit 1 for points that are "normal w.r.t. the standard continued fraction.

Exercise 15. Let $(X, \mathcal{B}, mu, f)$ be an ergodic probability m.p.t. Use the Birkhoff ergodic theorem to show that the following variation of Kac's Lemma holds: $\int_X \tau_A(x) d\mu = \mu(A)^{-1}$ for every $A \in \mathcal{B}$ with $\mu(A) > 0$ and $\tau_A : X \to \mathbb{N}$ is the first hitting time $\tau_A(x) = \min\{n \geq 1 : f^n(x) \in A\}$.

Exercise 16. Let $(X, \mathcal{B}, mu, f)$ be an infinite σ-finite ergodic m.p.t., and $A \in \mathcal{B}$ such that $\mu(A) < \infty$. Show that the ergodic average $\frac{1}{n} \sum_{j=0}^{n-1} 1_A \circ f^i(x)$ tends to 0 for μ-a.e. $x \in X$.

Exercise 17. Show that for each integer $n \geqslant 2$, the interval map given by

$$T_n(x) = \begin{cases} nx & \text{if } 0 \leqslant x \leqslant \frac{1}{n}, \\ \frac{1}{x} - \lfloor \frac{1}{x} \rfloor & \text{if } \frac{1}{n} < x \leqslant 1, \end{cases}$$

has invariant density $h(x) = \frac{1}{\log 2} \frac{1}{1+x}$.

Exercise 18. Show that the Perron-Frobenius operator has the following properties:

(1) P_T is linear;
(2) P_T is positive: $f \geq 0$ implies $P_T f \geq 0$.
(3) $\int P_T f \, d\mu = \int f \, d\mu$.
(4) $P_{T^k} = (P_T)^k$.

Exercise 19. Show that Lebesgue measure μ is mixing for the doubling map. In fact, for all dyadic intervals A, B there is $n_0 = n_0(A, B)$ such that the n-th **correlation coefficient**

$$\mathrm{Cor}_n(A, B) := \mu(T^{-n}(A) \cap B) - \mu(A)\mu(B) = 0$$

for every $n \geq n_0$.

Exercise 20. Show that $\|P_T f\|_{L^1(\mu)} \leq \|f\|_{L^1(\mu)}$, where P_T is the transfer operator w.r.t. $(X, \mathcal{B}, \mu; T)$.

Exercise 21. Suppose we have two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ satisfying: there are $\sigma \in (0, 1)$ and $L > 0$ such that:

$$a_{n+1} \leq a_n \quad \text{and} \quad b_n \leq \sigma b_n + L a_n.$$

Show that $(b_n)_{n\in\mathbb{N}}$ is bounded. Show that $\limsup_n b_{n+1} \leq L \sup_k a_k/(1-\sigma)$.

Exercise 22. Let $(\Sigma, \mathcal{B}, \mu_p)$ be a Bernoulli shift with probability vector $p = (p_1, \dots, p_N)$. Show that the left-shift σ preserves μ_p, both in the two-sided and one-sided case.

Exercise 23. Let $(\Sigma, \mathcal{B}, \mu_p)$ be a Bernoulli shift with probability vector $p = (p_1, \dots, p_N)$.

- Show that the set of periodic points of $(\Sigma, \mathcal{B}, \mu_p; \sigma)$ has measure zero if and only if $p_i < 1$ for all i.
- Show that the set of points in $(\Sigma, \mathcal{B}, \mu_p; \sigma)$ with a dense orbit has measure one if and only if $p_i > 0$ for all i.

Exercise 24. The 2-sided $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ shift is isomorphic to the 2-sided $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2})$-shift, but to no other 2-sided shift on ≤ 4 symbols.

Exercise 25. Show that the baker map with Lebesgue measure is isomorphic to the two-sided $(\frac{1}{2}, \frac{1}{2})$ Bernoulli shift.

Exercise 26. Show that every rational point $x \in \mathbb{T}^2$ is periodic under Arnol'd's cat map T_C. Explain why this implies that the cat returns.

Exercise 27.
- Where is hyperbolicity used in this proof of "For every hyperbolic toral automorphism, Lebesgue measure is ergodic and mixing"?
- Show that $A^n v = v$ for some vector $v \neq 0$ if and only if A has an eigenvalue α which is a root of unity (i.e., $\alpha^q = 1$ for some $q \in \mathbb{N}$).
- Show that also a non-hyperbolic toral automorphism ergodic if it has no eigenvalues that are root of unity.

Exercise 28. Show that the circle rotation is not mixing. Show that it is not weak mixing either.

Exercise 29. Let T_s be the tent map with slope $s \in (1, 2]$ (i.e., $T_s(x) = \min\{sx, s(1-x)\}$ for $x \in [0, 1]$).

- Show that the topological entropy $h_{top}(T_s) = \log s$.
- T_s has an absolutely continuous measure μ_s. Use Rokhlin's formula to show that μ_s is a measure of maximal entropy.

Exercise 30. Show that the Gauß map $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ has infinite topological entropy.

Exercise 31. A subset $X \in \{0,1\}^{\mathbb{N}}$ is called a subshift if it is closest (in product topology - or equivalently, in the metrix $d(x,y) = 2^{-n}$ for $n = \min\{k \geq 0 : x_k \neq y_k\}$) and shift-invariant, i.e., $x = x_0x_1x_2x_3 \cdots \in X$ implies $\sigma(x) = x_1x_2x_3 \cdots \in X$. The entropy of this subshift is

$$h_{top}(ZX, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log p_n(X) \qquad p_n(X) = \#\{w \in \{0,1\}^n : w \text{ appears as subword in some } x \in X\}.$$

- Show that this formula corresponds to Bowen's definition of entropy.
- Show that this limit indeed exists.

Exercise 32. Let (X, a) be the dyadic odometer and define $v_{m,n} : \{0,1\}^{\mathbb{N}} \to \mathbb{C}$ as

$$v_{m,n}(x) = e^{2\pi i m k/2^n} \qquad \text{if } x \in [a_1 \dots a_n],\ k = \sum_{j=1}^{n} a_j 2^{j-1}.$$

- Verify that the eigenfunctions form a orthonormal system.
- Show that the dyadic adding machine is uniquely ergodic.