Exercises for the Proseminar Stochastic Processes
Winter Semester 2018-19, PS250069

Exercise 1 *Compute explicitly the moment-generating and characteristic functions of the Gaussian distribution* $\mathcal{N}(\mu, \sigma^2)$ *and the Poisson distribution* $Pois_\lambda$.

Exercise 2 *Let* X, Y *be independent random variables. Use characteristic functions (or the moment-generating function) to compute the distribution of* $X + Y$ *if*

- $X \simeq \mathcal{N}(\mu_1, \sigma_1^2)$ *and* $Y \simeq \mathcal{N}(\mu_2, \sigma_2^2)$;
- $X \simeq Pois_{\lambda_1}$ *and* $Y \simeq Pois_{\lambda_2}$.

Exercise 3 *The referee at a baseball game has to do a toss at the beginning of the game to decide which team is going to bowl first. However, he cannot be sure that he has a fair coin. What can he do with this potentially unfair coin to produce a fair decision? What is the expected number of coin tosses to come to this decision.*

Exercise 4 *Consider random variables* $X_\lambda \simeq Pois_\lambda$. *Using characteristic functions, show that* $\frac{X_\lambda - \lambda}{\sqrt{\lambda}} \to^d Y$ *where* Y *is a standard normally distributed random variable.*

Exercise 5 (Example of how Kolmogorov's Extension Theorem works) *Let* $\Omega = \{(\omega_n)_{n\geq 0} : \omega_n \in \{-1, 1\}\}$ *be the probability space of a stochastic process of coin-flips with a fair coin. Let* $[e_0 \dots e_{k-1}] = \{(\omega_n)_{n\geq 0} \in \Omega : \omega_n = e_n,\ 0 \leq n < k\}$ *be a* k*-cylinder. Since the coin is fair,* $\mathbb{P}([e_0 \dots e_{k-1}]) = (\frac{1}{2})^k$ *for every choice of* $e_0, \dots, e_{k-1} \in \{-1, 1\}$.

Let

$$H_{k,m} = \bigcup \Big\{ [e_0, \dots, e_{k-1}] : |\frac{1}{k}\#\{0 \leq i < k : e_i = -1\} - \frac{1}{2}| \leq \frac{1}{m} \Big\}.$$

1. *Let* $\mathcal{B}$ *be the* σ*-algebra generated by all cylinder sets. Show that*

$$L = \{(\omega_n)_{n\geq 0} \in \Omega : \lim_{k\to\infty} \frac{1}{k}\#\{0 \leq i < k : \omega_i = -1\} = \frac{1}{2}\}$$

   belongs to $\mathcal{B}$.

2. *Show that* $\mathbb{P}(L) = 1$.

Solution to Exercise 5: 1. Clearly $H_{k,m}$ is a finite union of cylinders, and it belongs to $\mathcal{F}$.

$$\begin{aligned} L &= \{(\omega_n)_{n\geq 0} : \forall m \in \mathbb{N}\, \exists k_0 \in \mathbb{N}\, \forall k \geq k_0\, |\frac{1}{k}\#\{0 \leq i < k : \omega_i = -1\} - \frac{1}{2}| < \frac{1}{m}\} \\ &= \bigcap_m \bigcup_{k_0 \in \mathbb{N}} \bigcap_{k \geq k_0} H_{k,m} \end{aligned}$$

is produced by countable intersections and countable unons, and therefore $L \in \mathcal{F}$.
2. Using $S_k = \#\{0 \leq i < k : \omega_i = -1\}$ and the CLT (with $\sigma^2 = \frac{1}{4}$) we have

$$\mathbb{P}(H_{k,m}^c) = \mathbb{P}(|\frac{1}{k}S_k - \frac{1}{2}| \geq \frac{1}{m}) = 2\mathbb{P}(\frac{1}{\sqrt{k}}(S_k - \frac{k}{2}) \geq \frac{\sqrt{k}}{m})$$

By the Central Limit Theorem, this tends to $2\int_{\frac{\sqrt{k}}{m}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{u^2}{2\sigma^2}}\,du$ as $k \to \infty$, so there is some $C > 0$ such that

$$\begin{aligned}
\mathbb{P}(H_{k,m}^c) &\leq 2C\int_{\frac{\sqrt{k}}{m}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{u^2}{2\sigma^2}}\,du \leq 2C\frac{m}{\sqrt{2\pi k}}\int_{\frac{\sqrt{k}}{m}}^{\infty} \frac{u}{\sigma} e^{-\frac{u^2}{2\sigma^2}}\,du \\
&= \frac{4C}{m\sqrt{2\pi k}}[-\sigma e^{-\frac{u^2}{2\sigma^2}}]_{\frac{\sqrt{k}}{m}}^{\infty} = \frac{4mC\sigma}{\sqrt{2\pi k}} e^{-\frac{k}{2m^2\sigma^2}}.
\end{aligned}$$

Set $\hat{H}_{k_0,m} = \bigcap_{k \geq k_0} H_{k,m}$. Then

$$\begin{aligned}
\mathbb{P}(\hat{H}_{k_0,m}) &= \mathbb{P}(\cap_{k\geq k_0} H_{k,m}) = \mathbb{P}(\Omega \setminus \cup_{k\geq k_0} H_{k,m}^c) \\
&\geq 1 - \sum_{k \geq k_0} \mathbb{P}(H_{k,m}^c) \geq 1 - \frac{4mC\sigma}{\sqrt{2\pi}} \sum_{k\geq k_0} \frac{1}{\sqrt{k}} e^{-\frac{k}{2\sigma^2 m^2}},
\end{aligned}$$

and this tends to 1 as $k_0 \to \infty$. Since $\hat{H}_{k_0,m}$ is increasing in k_0 and $\bigcup_{k_0\in\mathbb{N}} \hat{H}_{k_0,m}$ is decreasing in m, we have

$$\mathbb{P}(L) = \mathbb{P}(\cap_m \cup_{k_0\in\mathbb{N}} \hat{H}_{k_0,m}) = \lim_{m\to\infty}\lim_{k_0\to\infty} \mathbb{P}(\hat{H}_{k_0,m}) = 1.$$

Exercise 6 *Consider the difference equation*

$$y_{n+2} + ay_{n+1} + by_n = c, \qquad n \in \mathbb{N},\ y_n \in \mathbb{R},\ a, b, c \in \mathbb{R}.$$

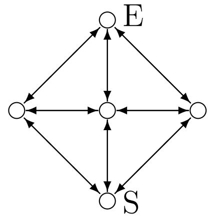
1. *Find the full solution if $a = -2$, $b = -3$, $c = 0$ and initial condition $y_0 = y_1 = 1$.*
2. *Find the full solution if $a = -2$, $b = -3$, $c = 2$ and initial condition $y_0 = y_1 = 1$.*
3. *Find the full solution if $a = -2$, $b = 1$, $c = 0$ and initial condition $y_0 = 1, y_1 = 2$.*

Exercise 7 *Let $A = (a_{ij})$ be an $n \times n$ probability matrix, i.e., $a_{ij} \geq 0$ and the row-sums $\sum_j a_{ij} = 1$. Show that*

1. *A^k is a probability matrix for every $k \in \mathbb{N}$;*
2. *1 is an eigenvalue of A;*
3. *there are no eigenvalues λ with $|\lambda| > 1$;*
4. *the left and right eigenvectors with eigenvalue 1 can be chosen such that $v_j \geq 0$ for all $j = 1, \dots, n$.*

Exercise 8 *Consider the following transition graph, where from each node, each passage to a neighbouring node is equally likely.*

1. *Give the transition probability matrix associated to this graph.*
2. *Starting from node S, what is the probability of returning to S after three, resp. four steps?*



3. *A marmot starts in S and walks the graph until she reaches state E. What is the probability that she returns to S before reaching E?*

4. *What is the expected number of steps for the marmot to reach the end?*

Exercise 9 *Lisa and Bart play a game with one die and six marbles. They start with three marbles each, and roll the die. If the number of spots is 1 or 2, then Bart loses two marbles to Lisa. If the number of spots is 3,4,5 or 6, then Lisa loses one marble to Bart. They continue until one of them has all the marbles, who then of course is the winner.*

1. *What is the probability that Lisa wins?*

2. *What is the probability that Lisa wins after first getting down to one marble?*

Exercise 10 *Given is the transition matrix*

$$P = (p_{ij})_{i,j=1}^{4} = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

1. *What are the communication classes?*

2. *Compute the expected return time μ_k for each state.*

3. *For which i, j holds: $\lim_{n\to\infty} p_{i,j}^{(n)} \to 1/\mu_j$?*

Exercise 11 *Given a Markov chain on a state space E, let $P = (p_{ij})_{i,j\in E}$ denote a transition matrix, and $P^n = (p_{ij}^{(n)})_{i,j\in E}$ its n-th power. Show the following:*

1. *If C is a communicating class of a finite Markov chain with $p_{ii} > 0$ for some $i \in \mathbb{C}$, show that there is $n \in \mathbb{N}$ such that $p_{ij}^{(n)} > 0$ for all $i, j \in C$.*

2. *If C is a closed communicating class, then $(p_{ij})_{i,j\in C}$ is a probability matrix.*

3. *Every finite Markov chain has at least one closed communicating class.*

4. *Find an example of a Markov chain without closed communicating class.*

Exercise 12 *A salmon has to jump up three waterfalls to reach the place where she spawns. Every attempt to jump a river succeeds with probability $\frac{4}{5}$, except that jumping the first waterfall will always succeed, but there is a chance of $\frac{1}{5}$ that she inadvertently falls down the previous watervall. But once up the river she spawns and dies (nature is hard).*

1. *Give a Markov chain description of this story. What are the communication classes? Which are closed?*

2. *What is the expected time, in number of waterfalls crossed (up or down), that the whole journey takes.*

3. *Supposing that every succeeded jump causes a 10gr weight loss. What is the expected weight loss at the end of the journey?*

4. *Same as the previous, but now the middle watervall is extra hard and causes a 30gr weight loss.*

Exercise 13 *Let* $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$ *be the transition matrix of a Markov chain on state space* $E = \{1, 2\}$. *Compute the expected return time* $\mu_i = \mathbb{E}(T_i)$ *for* $i = 1, 2$. *What is the distribution of* T_i*?*

Exercise 14 *A bag contains* N *red and green balls. We draw balls, and whichever colour we find, we put the ball back together with another ball of the same colour. Let* R_n *be the number of red balls in the bag after the* n*-th drawing.*

1. *Show that* $(\frac{R_n}{N+n})_{n\geq 1}$ *is a martingale.*

2. *Use the Optional Stopping Theorem to compute the expectation* $\mathbb{E}(\frac{1}{T+N})$ *if* $T \geq 0$ *is the number of drawings when the first green ball is drawn.*

Exercise 15 *A chess-king moves on a* 3×3 *chess-board, taking each of its possible moves (a king can move to each of the neighbouring squares, horizontally, vertically or diagonally) with equal probability. What are the limit visit frequencies to each of the squares?*

You can try for the other chess-pieces too.

Exercise 16 *Consider a Markov chain and recall the notation*

$$p_{ij}^{(n)} = \mathbb{P}(X_n = j, X_0 = i), \quad f_{ij}^{(n)} = \mathbb{P}(X_n = j, X_0 = i, X_k \neq i, j \text{ for } 0 < k < n\}$$

Also recall that i *is a* recurrent state *if* $\sum_{n\geq 1} f_{ii}^{(n)} = 1$ *and a* positive recurrent state *if* $\sum_{n\geq 1} n f_{ii}^{(n)} < \infty$.

1. *Show that* $\sum_{n\geq 1} f_{ii}^{(n)} = 1$ *is equivalent to* $\sum_{n\geq 1} p_{ii}^{(n)} = \infty$.

2. *Show that all states in a communication class are simulatenously (null/positive recurrent or transient).*

3. *Show that if the Markov chain is irreducible and finite, then every state is positively recurrent.*

Solution to Exercise 16: Part 1. $\sum_{n\geq 1} f_{ii}^{(n)} = \mathbb{P}(T_i < \infty | X_0 = i)$ for the stopping time $T_i = \min\{n : X_n = i\}$. If $\sum_{n\geq 1} f_{ii}^{(n)} = 1$, then by the strong Markov property, also $\mathbb{P}(T_i^k < \infty) = 1$ for all $k \geq 1$, for the successive return times $T_i^k = \min\{n > T_i^{k-1} : X_n = i\}$, $T_i^1 = T_i$. Therefore $\mathbb{P}(X_n = i \text{ infinitely often}) = 1$, and hence $\sum_{n\geq 1} p_{ii}^{(n)} = \infty$. Conversely, if $\sum_{n\geq 1} f_{ii}^{(n)} = \gamma < 1$, then by the strong Markov property $\mathbb{P}(T_i^k < \infty | T_i^{K-1} < \infty) = \gamma$, and hence $\mathbb{P}(X_n = i \text{ at least } k \text{ times}) = \gamma^k$, so $\sum_{n\geq 1} p_{ii}^{(n)} \geq k$ occurs with probability γ^k. In the limit $\mathbb{P}(X_n = i \text{ infinitely often}) = \mathbb{P}(\sum_{n\geq 1} p_{ii}^{(n)} \geq \infty) = 0$.
Part 2. If i, j belong to the same communication class, then there is $M, N \in \mathbb{N}$ such that $p_{ij}^{(M)}, p_{ji}^{(N)} > 0$. Since $p_{ii}^{(n+M+N)} \geq p_{ij}(M) p_{jj}^{(n)} p_{ji}^{(N)}$, we have that $\sum_n p_{ii}^{(n)}$ and $\sum_n p_{jj}^{(n)}$ converge/diverge simulateneously. So both states are transient or both states are recurrent.

The argument for null-recurrence is a bit more involved. Assume that state i is positive recurrent, so the expected return time $\mu_i := \mathbb{E}(T_i | X_0 = i) < \infty$. Recall from the lecture that

$$\gamma^j := \mathbb{E}(\#\{0 \leq k < T_i : X_k = j\} | X_0 = i\}$$

satisfies $\gamma P = \gamma$, $\gamma^i = 1$ and $\sum_j \gamma^j = \mu_i < \infty$. Therefore $\pi = \gamma/\mu_i$ is a normalized positive left-eigenvector of the transtion matrix P. Moreover, since the MC is irreducible, for each j there is a N_j minimal such that $p_{ij}^{(N_j)} > 0$, so that $\gamma^j \geq \pi_{ij}^{(N_j)} > 0$, and therefore also $\pi_j > 0$ for each j. Also from the lecture: $\mu_j = \mathbb{E}(T_j : X_0 = j) = 1/\pi_j < \infty$, so state j is positive recurrent. This showsthat if one state is positive recurrent, then they all are, and hence if one state is null-recurrent, they all are.
Part 3. Because the MC is irreducible, we can find N_j such that $p_{ji}^{(N_j)} > 0$ and since the MC is finite, $N := \max_j N_j < \infty$ and $p = \min_j p_{ji}^{(N_j)} > 0$. This means that $f_{ii}^{(n)} \leq 1 - p$ for $N \leq n < 2N$. By induction $f_{ii}^{(kN)} \leq (1-p)^k$ for $kN \leq n < kN + N$, so $\sum_{(n)} f_{ii}(N) \leq N \sum_k (1-p)^k < \infty$. Similarly $\sum_{(n)} n f_{ii}(n) \leq N \sum_k k(1-p)^k < \infty$. Hence i is a positive recurrent.

Exercise 17 *Given a finite irreducible Markov chain, show that all states are not just recurrent, but* **positive** *recurrent, i.e.,* $\mu_i = \mathbb{E}(T_i) < \infty$ *for the return time* T_i *to state* i*, and any* $i \in E$*. Show that* T_i *has finite variance as well.*

Exercise 18 *Let* $(X_n)_{n\geq 0}$ *be a random walk on the non-negative integers, with* $p_{0,1} = 1$ *and* $p_{n,n-1} = p$*,* $p_{n,n+1} = 1 - p$ *for* $n \geq 1$*. For which* p *is this random walk positive recurrent/null-recurrent/transient? What is the stationary distribution is the Markov chain is positive recurrent? Is there a stationary distribution if the Markov chain is null-recurrent?*

Exercise 19 *Let* G *be an infinite rooted binary graph, i.e., there is a single root* R *and every other vertex has exactly three neighbours (so two "below" it), and there are no loops. Let* $(X_n)_{n\geq 0}$ *be a random walk on* G*.*

1. *Suppose, in one step you can only go to a neighbouring vertex, and from every vertex there is equal probability to jump to any of its neighbours. Is this random walk positive recurrent/null-recurrent/transient?*

2. *Same question, but now the probability to jump to the neighbour closer to the root is* $\frac{1}{2}$ *(and from the root you always jump to its single neighbour).*

Exercise 20 *Let P be the transition matrix of a finite state Markov chain, and assume that P is irreducible, but periodic with period $d \geq 2$. Show that state space E decomposes into d communication classes for the d-th iterate of the process, i.e., for the process $(X_{dn})_{n\geq 0}$. Show that $e^{2\pi ic/d}$ is an eigenvalue of P for all integers $0 \leq c < d$.*

Exercise 21 *We are given an irreducible aperiodic Markov chain with finite state space E, transition matrix P and stationary distribution π.*

1. *Show that the first return process to a subset E' is also a Markov chain; what is the stationary distribution of this Markov chain?*

2. *If there are states $i, i' \in E$ such that $p_{ij} = p_{i'j}$ and $p_{ji} = p_{ji'}$ for all $j \in E$, show that we find a new Markov chain by merging states i and i'; what is the stationary distribution of this Markov chain?*

Exercise 22 *Let $(X_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}$, with transition probabilities $p_{n,n-2} = \frac{1}{4}$, $p_{n,n-1} = \frac{1}{4}$ and $p_{n,n+1} = \frac{1}{2}$. Is this random walk positive recurrent/null-recurrent/transient?*

Exercise 23 *Let $(X_n)_{n\geq 0}$ be a symmetric random walk on $\mathbb{Z}$ with $X_0 = 0$. Let $u_{2n} = \mathbb{P}(X_{2n} = 0)$ and $f_{2n} = \mathbb{P}(\min\{k \geq 1 : X_{2k} = 0\} = n)$.*

1. *Show that $f_{2n} = \frac{1}{2n-1} u_{2n}$.*
2. *Show that $u_{2n} = \sum_{k=1}^{n} f_{2k} u_{2(n-k)}$.*
3. *Show that $\sum_{k=0}^{n} u_{2k} u_{2(n-k)} = 1$.*

Exercise 24 *The Taqqus of the Planet Koozebane*[1] *each have $K \in \mathbb{N}$ offspring before they evaporate. For each Taqqu, K is independent of everything else and has distribution $\mathbb{P}(K = 0) = \mathbb{P}(K = 1) = \frac{1}{4}$, $\mathbb{P}(K = 2) = \frac{1}{2}$.*

1. *Compute the generating function and the moment generating function of K.*

2. *Assume that X_n is the total population of Taqqus, starting with $X_0 = 1$. Compute the probability that the Taqqus go extinct.*

3. *Show that, provided the Taqqus survive for n steps, that the probability that they die out decreases to zero, superexponentialy in n.*

Exercise 25 *Let $N_t \simeq Pois(\lambda t)$. Show that $\mathbb{P}(N_{t+h} = 0) = (1 - \lambda h - o(\lambda h))\mathbb{P}(N_t = 0)$. If we write $p(t) = \mathbb{P}(N_t = 0)$, show that $p(t)$ satisfies the differential equation $p'(t) = \lambda p(t)$. Solve this equation.*

Exercise 26 *Telephone calls arrive at the station according to a Poisson process with an hourly rate λ.*

1. *The phone equipment is not entirely functioning: a phonecall is not properrly connected with probability q. Show that the number of properly received calls has distribtuion $Pois(\lambda(1-q)t)$ for time unit an hour.*

2. *A second stream of phone calls comes in with hourly rate μ. Find the distribution of total number of incoming calls. After putting down the phone, how much time does the operator have to wait on average for the next call?*

[1] https://www.youtube.com/watch?v=vbXzpoH6m2c