Forward and Backward Equations of a Counting Process

Given a stationary counting process $N(t)$ with only positive increases (i.e., a birth process), let

$$p_{ij}(t) = \mathbb{P}(N(t) = j | N(0) = i) = \mathbb{P}(N(t+s) = j | N(s) = i).$$

Let $\mathbb{P}(N(t+h) = j | N(t) = j) = 1 - \lambda_j h + o(h)$ and $\mathbb{P}(N(t+h) = j+1 | N(t) = j) = \lambda_j h + o(h)$ and $\mathbb{P}(N(t+h) \geq j+2 | N(t) = j) = 1 - \lambda_j h + o(h)$. Then, neglecting an increase of ≥ 2 in a time interval $[t, t+h]$,

$$\begin{aligned}
p_{ij}(t+h) &= \frac{\mathbb{P}(N(t+h) = j \wedge N(0) = i)}{\mathbb{P}(N(0) = i)} \\
&= \frac{\mathbb{P}(N(t+h) = j \wedge N(0) = i \wedge N(t) = j) + \mathbb{P}(N(t+h) = j \wedge N(0) = i \wedge N(t) = j-1)}{\mathbb{P}(N(0) = i)} \\
&= \frac{\mathbb{P}(N(t) = j \wedge N(0) = i)}{\mathbb{P}(N(0) = i} \frac{\mathbb{P}(N(t+h) = j \wedge N(0) = i \wedge N(t) = j \wedge N(0) = i)}{\mathbb{P}(N(t) = j \wedge N(0) = i)} \\
&\quad + \frac{\mathbb{P}(N(t) = j-1 \wedge N(0) = i)}{\mathbb{P}(N(0) = i} \frac{\mathbb{P}(N(t+h) = j \wedge N(0) = i \wedge N(t) = j-1 \wedge N(0) = i)}{\mathbb{P}(N(t) = j-1 \wedge N(0) = i)} \\
&= \mathbb{P}(N(t) = j | N(0) = i) \mathbb{P}(N(t+h) = j | \mathbb{P}(N(t) = j \wedge N(0) = i) \\
&\quad + \mathbb{P}(N(t) = j-1 | \mathbb{P}(N(0) = i) \mathbb{P}(N(t+h) = j | \mathbb{P}(N(t) = j-1 \wedge N(0) = i) \\
&= \mathbb{P}(N(t) = j | N(0) = i) \mathbb{P}(N(t+h) = j | \mathbb{P}(N(t) = j) \\
&\quad + \mathbb{P}(N(t) = j-1 | \mathbb{P}(N(0) = i) \mathbb{P}(N(t+h) = j | \mathbb{P}(N(t) = j-1) \\
&= p_{i,j}(t)(1 - \lambda_j h + o(h)) + p_{i,j-1}(t)(\lambda_{j-1} h + o(h)).
\end{aligned}$$

In the penultimate line above, we used the Markov property (so $\mathbb{P}(N(t+h) = j | \mathbb{P}(N(t) = j-1 \wedge N(0) = i) = \mathbb{P}(N(t+h) = j | \mathbb{P}(N(t) = j-1)$ etc., because what happens at time $t+h$ depends on what happened at time t, but not how you got to what happened at time t.

Rearranging gives

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t) + o(1).$$

Now take the limit as $h \to 0$. This gives the **forward equations**:

$$\begin{cases} p'_{ij}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{i,j}(t) & \text{if } j \geq i, \\ p_{ij}(0) = \delta_{ij} & \text{(Kronecker } \delta\text{)}, \\ p_{ij}(t) \equiv 0 & \text{if } j < i. \end{cases}$$

In a slightly different way, and neglecting an increase of ≥ 2 in a time interval $[0, h]$,

$$\begin{aligned}
p_{ij}(t+h) &= \frac{\mathbb{P}(N(t+h)=j \wedge N(0)=i)}{\mathbb{P}(N(0)=i)} \\
&= \frac{\mathbb{P}(N(t+h)=j \wedge N(0)=i \wedge N(h)=i)}{\mathbb{P}(N(0)=i)} \\
&\quad + \frac{\mathbb{P}(N(t+h)=j \wedge N(0)=i \wedge N(h)=i+1)}{\mathbb{P}(N(0)=i)} \\
&= \frac{\mathbb{P}(N(t+h)=j \wedge N(h)=i \wedge N(0)=i)}{\mathbb{P}(N(h)=i \wedge N(h)=0)} \frac{\mathbb{P}(N(h)=i \wedge N(0)=i)}{\mathbb{P}(N(0)=i)} \\
&\quad + \frac{\mathbb{P}(N(t+h)=j \wedge N(h)=i+1 \wedge N(0)=i)}{\mathbb{P}(N(h)=i+1 \wedge N(0)=i)} \frac{\mathbb{P}(N(h)=i+1 \wedge N(0)=i)}{\mathbb{P}(N(0)=i)} \\
&= \mathbb{P}(N(t+h)=j | N(h)=i \wedge N(0)=i)\mathbb{P}(N(h)=i | N(0)=i) \\
&\quad + \mathbb{P}(N(t+h)=j | N(h)=i+1 \wedge N(0)=i)\mathbb{P}(N(h)=i+1 | N(0)=i) \\
&= \mathbb{P}(N(t+h)=j | N(h)=i)(1-\lambda_i h + o(h)) \\
&\quad + \mathbb{P}(N(t+h)=j | N(h)=i+1)(\lambda_i h + o(h)) \\
&= p_{i,j}(t)(1-\lambda_i h + o(h)) + p_{i+1,j}(t)(\lambda_i h + o(h)).
\end{aligned}$$

Again, we used the Markov property and stationarity in the last two lines.

Rearranging gives

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \lambda_i p_{i+1,j}(t) - \lambda_i p_{i,j}(t) + o(1).$$

Now take the limit as $h \to 0$. This gives the **backward equations**:

$$\begin{cases} p'_{ij}(t) = \lambda_i p_{i+1,j}(t) - \lambda_i p_{i,j}(t) & \text{if } j \geq i, \\ p_{ij}(0) = \delta_{ij} & \text{(Kronecker } \delta), \\ p_{ij}(t) \equiv 0 & \text{if } j < i. \end{cases}$$