The (profinite) fundamental group of a tiling space

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Summer Topology Conference, Vienna

20 July 2022

Tiles, tilings, tiling spaces

- tiles: assume here polyhedra, meeting face-to-face
- tiling: covering of $\mathbb{R}^d$ by tiles, finitely many local patterns (FLC)
- tiling metric: two tilings are ε-close, if they agree in a ball $B_{1/\varepsilon}(0)$ after an ε-small translation
- tiling space: closed, tranlsation invariant space of tilings

Under mild assumptions, tiling spaces are compact.

Inflation tilings are self-similar, generated by inflation procedure:

1. scale tiling by factor λ
2. disect inflated tiles into tiles of original size

Tiling spaces are inverse limit spaces

$$\mathbb{X} = \varprojlim X_i = \left\{ (x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i \,\middle|\, x_i = \tilde{f}_i(x_{i+1}) \right\}$$

X_i approximants of $\mathbb{X}$: finite cell complexes, whose points represent cylinder sets of tilings

$\tilde{f}_i$ continuous cellular surjection; each $x_i \in X_i$ has typically several preimages under $\tilde{f}_i$

For inflation tilings, all X_i can be chosen equal, likewise for $\tilde{f}_i$. Points in X_1, X_2, ... represent cylinder sets of (collared) tiles, supertiles, etc.

Tiling spaces are locally Euclidean $\times$ Cantor, connected,

but not path-connected.

Topological invariants of tiling spaces

Čech cohomology behaves well under inverse limits:

$$\check{H}(\varprojlim X_i) = \varinjlim \check{H}(X_i) = \varinjlim H(X_i)$$

This is practically computable!

Other common invariants only see path-components – not interesting.

However, if $(x_i)_{i\in\mathbb{N}}$ is an inflation fixpoint, we have an inverse system

$$\pi_1(X_1, x_2) \xleftarrow{f_1} \pi_1(X_2, x_2) \xleftarrow{f_2} \pi_1(X_3, x_3) \xleftarrow{f_3} \cdots$$

of fundamental groups and homomorphisms between them. Is this useful?

Topological invariants of tiling spaces II

More promising are the dual objects $Hom(\pi_1(X_i, x_i), G)$, with G any group, for which there is a direct limit of the system

$$Hom(\pi_1(X_1, x_2), G) \xrightarrow{f_{1*}} Hom(\pi_1(X_2, x_2), G) \xrightarrow{f_{2*}} \cdots$$

This had already been proposed by Sadun. Erdin (arxiv:1002.1460) has expressed the represenation varieties $Hom(\pi_1(X_i, x_i), G)/G$ in terms of flat, principle G-bundles, and showed that their direct limit $\varinjlim Hom(\pi_1(X_i, x_i), G)/G$ is a homeomorphism invariant of $\mathbb{X}$.

Moreover, for G finite, this is practically computable!

Topological invariants of tiling spaces III

Here, we go a different route. We remark that with $F = \pi_1(X_i, x_i)$, also its profinite completion $\hat{F}$ is a good invariant of X_i.

Since $\hat{F}$ already is an inverse limit (a profinite group), the corresponding inverse limit $\varprojlim \hat{F}$ is much better behaved. In fact, it is again a profinite group, and a genuine invariant of $\mathbb{X}$.

Along with $\varprojlim \hat{F}$, also $\varinjlim Hom(\hat{F}, G)$ is an invariant, and is computable for G finite, because then,

$$\varinjlim Hom(\hat{F}, G) = \varinjlim Hom(F, G)$$

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Interlude: Profinite completion

Let F be a finitely generated group, and set

$\mathcal{N} = \{N_i\}_{i \in I} := \{\text{all finite-index normal subgroups of } F\}$,

$\mathcal{Q} = \{F/N_i\}_{i \in I} := \{\text{all finite quotients of } F\}$.

Let $\eta_i : F \to F/N_i$ be the canonical projection.

If $N_i \geq N_j$, there exist morphisms $\Phi_{ji} : F/N_j \to F/N_i$ with $\eta_i = \Phi_{ji} \circ \eta_j$.

If $N_i \geq N_j \geq N_K$, we have $\Phi_{ki} = \Phi_{ji} \circ \Phi_{kj}$, and for any $N_i, N_j \in \mathcal{N}$, there exists $N_k \in \mathcal{N}$ with $N_i \geq N_k$, $N_j \geq N_k$.

Interlude: Profinite completetion II

The quotients $\mathcal{Q}$ form a partially ordered, directed set, and we can form an inverse limit:

$$\hat{F} = \varprojlim F/N_i = \{(g_i)_{i\in I} | g_i \in F/N_i, g_i = \Phi_{ji}(g_j) \text{ whenever } F/N_i \leq F/N_j\}\,.$$

$\hat{F} \subset \prod_{i\in I} F/N_i$ is the profinite completion of F.

Giving F/N_i the discrete topology, $\hat{F}$ becomes a topological group.

There is a natural map $\eta : F \to \hat{F}$ given by $\eta(g) = (\eta_i(g))_{i\in I}$, which is injective iff $\cap_{i\in I} N_i$ is trivial.

$\hat{F}$ is the completion of F in the minimal topology in which all N_i are open.

Universality property of η

For every homomorphism $h : F \to H$ to a profinite group H, there exists a unique continuous $\hat{h} : \hat{F} \to H$ with $h = \hat{h} \circ \eta$. In particular, this holds for every H finite.

Given an endomorphism $f : F \to F$, there is a unique continuous endomorphism $\hat{f} : \hat{F} \to \hat{F}$ making the following diagram commutative:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\eta"'] & F \arrow[d, "\eta"'] \\ \hat{F} \arrow[r, "\hat{f}"] & \hat{F} \end{tikzcd}$$

The profinite fundamental group

Suppose we have a inflation tiling space constructed from a single approximant X of collared tiles, and assume $x_0 \in X$ is a fixpoint.

Then, we can replace the inverse system on the fundamental group $F = \pi_1(X, x_0)$ with endomorphism f by the corresponding system on the profinite completion $\hat{F}$ and the induced endomorphism $\hat{f}$:

$$\hat{F} \xleftarrow{\hat{f}} \hat{F} \xleftarrow{\hat{f}} \hat{F} \xleftarrow{\hat{f}} \dots$$

This system has a well-behaved inverse limit $\varprojlim \hat{F}$, the profinite fundamenatal group $\hat{\pi}_1(\mathbb{X}, x_0)$. It is a profinite group, and in fact a genuine homeomorphism invariant of $\mathbb{X}$.

Representation varieties

With $\hat{\pi}_1(\mathbb{X}, x_0) = \varprojlim \hat{F}$, also the represenation varieties $Hom(\hat{\pi}_1(\mathbb{X}, x_0), G) = \varinjlim Hom(\hat{F}, G)$, G any group, are invariants.

If G is finite, $Hom(F, G)$ is finite, and due to the universality property of η, $|Hom(F, G)| = |Hom(\hat{F}, G)|$ (here, $Hom(\hat{F}, G)$ consists of continuous homomorphisms). Note that $h : F \to G$ factors uniquely through $\hat{F}$.

Under the endomorphism f of F, the homomorphism $h : F \to G$ is mapped to $h \circ f$. For any finite G, $\varinjlim Hom(F, G)$ is a finite set of homomorphisms. Its size is an invariant of $\mathbb{X}$.

Fundamental group of a cell comples – 1d

Computing F of a 1d complex:

- inscribe spanning tree S in X
- $\pi_1(X, x_0) = \pi_1(X, S, x_0)$
- For each 1-cell $c \in X$, we have a free group generator g_c. We have:

  $F = \langle g_c \rangle_{c \in X \setminus S} = \langle g_c \rangle_{c \in X} / [\langle g_c \rangle_{x \in S}]_{nc}$

To each "bridge" $c \in X \setminus S$, there corresponds a unique loop at x_0, and an element in $\tilde{F} = \langle g_c \rangle_{c \in X}$. We take its image under inflation and mod out non-bridge generators, to obtain the image of a bridge generator in F.

Fundamental group of a cell comples – dim ≥ 2

First, construct π_1 of the 1-skeleton.

Then, each 2-cell provides a loop homotopic to zero. The path-ordered product $\prod_{c' \in \partial c} g_{c'}$ is trivial in π_1, gives a relation. For all 2-cells, mod out those relations from π_1 of 1-skeleton.

In $d \geq 2$, $\pi_1(X, x_0)$ is not free in general, but finitely presented. This is no fundamental obstacle, but makes computation a bit more complicated.

In $d = 1$, $\pi_1(X, x_0)$ is always a free group.

Remarks

- if $f : F \to F$ is an isomorphism, $\varprojlim \hat{F} = \hat{F} \to$ we cannot distinguish different isomorphisms
- if G is abelian, $Hom(\varprojlim \hat{F}, G)$ is not more informative than $H^1(\mathbb{X}, G) \to$ better take G non-abelian
- if F is free, $f(F)$ is also free, possibly of smaller rank; first iterate f until rank no longer shrinks, and work wit $f^k(G)$
- abelianisation of $\varprojlim F$ gives 1-cohomology, so $\mathrm{rk} H^1 \leq \mathrm{rk} F$ $\mathrm{rk} F$ can be bigger if abelianisation of f contains eigenvalues 0 (e.g. Thue-Morse)

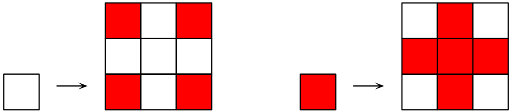
1d Examples

Inflations with matrix $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$. Taking all possible permutations of tiles in the supertiles, we get 20 different inflations, which fall into 2 symmetric and 4 mirror pairs of MLD classes. These 6 classes can all easily be distinguished with small groups G, like S_3 or S_4.

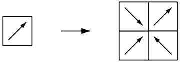
Inflations with matrix $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. Here, we get 30 inflations falling into 2 symmetric and 5 mirror pairs of MLD classes, which can be distinguished.

Many other cases have been computed, with similar results. The only problematic cases are those with several isomorphisms f of the same rank, which can't be distinguished.

2d Examples I



For the squiral tiling, F is free abelian of rank 2, and has the same $\hat{\pi}_1$ as the 2d 3-adic solenoid, very disappointing...



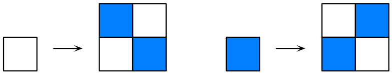
For the chair tiling, F is not abelian, but almost. For most small groups,

$Hom(\hat{\pi}_1(\text{chair}), G) = Hom(\hat{\pi}_1(\text{2d 2-adic solenoid}), G)$.

But there are two groups of order 27 where they differ.

For tilings with simple H^1, F tends to be too simple: abelian, eventually abelian, almost abelian,...

2d Examples II



For the 2d Thue-Morse tiling, 1-cohomology is more complicated, and so are F (of rank 4) and the endomorphism acting on it. Here we get results which allow to distinguish it from other tilings

For the octagonal Ammann-Beenke tiling, F has rank 5 (like H^1), and also gives non-trivial results.

So, we need a sufficiently rich H^1 for a rich invariant $Hom(\hat{\pi}_1, G)$.