Compact Generators

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Generators

Let X be a T_1 space.

Recall X is Tychonoff if:

whenever x is not in closed C

then there is a g in $C(X)$ such that $g(x) \notin \overline{g(C)}$

Here, $C(X) =$ all continuous real-valued functions on X.

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Lemma

A space X is Tychonoff if and only if

$$\{g^{-1}U : g \in C(X), U \text{ open in } \mathbb{R}\} \text{ is a base for } X.$$

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Definition

A subset G of $C(X)$ is a generator if:

whenever x is not in closed C,

then there is a g in G such that $g(x) \notin \overline{g(C)}$

Lemma

A subset G of $C(X)$ is a generator if and only if

$\{g^{-1}U : g \in G, U \text{ open in } \mathbb{R}\}$ *is a base for X.*

Natural Topologies on $C(X)$

Give $C(X)$ the pointwise convergence topology.

Denote by $C_p(X)$.

Basic open neighborhoods of f have form:

$$B(f, F, \epsilon) = \{g \in C(X) : \forall x \in F \ |f(x) - g(x)| < \epsilon\},$$

where F is a finite subset of X, and $\epsilon > 0$.

Give $C(X)$ the compact-open topology.

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Any generator $G \subseteq C(X)$ is a subspace of $C_p(X)$ and of $C_k(X)$.

Metrizable generator, *separable generator*, *discrete*, *compact*, σ-*compact*, or *Lindelöf generator*...

Compact Generators

Note 1: if $C_k(X)$ (σ-)compact then $C_p(X)$ is (σ-)compact.
Note 2: $C_p(X)$ is never compact.

Theorem (Velichko)

$C_p(X)$ is σ-compact if and only if X is finite.

Tkachuk & Shakmatov: σ-countably compact.
Reznichenko: σ-pseudocompact.

Which spaces have a compact generator:

1. In $C_k(X)$?
2. In $C_p(X)$?
3. σ-compact?

Which spaces have a countably compact or pseudocompact generator in $C_p(X)$ or $C_k(X)$?

Some Notation

$D(S) =$ the set S with discrete topology.

$\alpha(X) =$ the one-point compactification of X.

Examples:

$$\alpha(D(\kappa)), \qquad \bigoplus_n \alpha(D(\kappa))^n, \qquad \alpha\left(\bigoplus_n \alpha(D(\kappa))^n\right).$$

Precise Generators

A space X is Tychonoff

$\iff$ whenever x is not in closed C
then there is a g in $C(X)$ such that $g(x) \notin \overline{g(C)}$

$\iff$ whenever x is not in closed C
then there is a g in $C(X)$ such that $g(C) \subseteq \{0\}$ but $g(x) \neq 0$

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A subset G of $C(X)$ is a

$(0, \neq 0)$-generator if: whenever x is not in closed C
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$(0, 1)$-generator if: whenever x is not in closed C
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Question inflation – now have $2 \times 2 \times 3 = 12$!

Generator Type Matters

Proposition

The following are equivalent:

(i) *X has a compact $(0, 1)$-generator in $C_k(X)$,*
(ii) *X has a compact $(0, 1)$-generator in $C_p(X)$, and*
(iii) *X is discrete.*

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X discrete: set $G = \{\chi_{\{x\}} : x \in X\} \cup \{\mathbf{0}\}$.

Clearly a $(0,1)$-generator for X. And in $C_k(X)$ see $G \equiv \alpha(D(X))$.

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Clearly a $(0,1)$-generator for X. And in $C_k(X)$ see $G \equiv \alpha(D(X))$.

X not discrete, say x_0 not isolated: set $f = \chi_{\{x_0\}} \notin C(X)$.
If G is any $(0,1)$-generator for X in $C_p(X)$, then f is in the pointwise closure of G, so G cannot be compact.

Take any basic nbd $B(f, F, \epsilon)$ of f in $\mathbb{R}^X$, and let $F' = F \setminus \{x_0\}$.

There is a g in G such that $g(F') \subseteq \{0\}$ but $g(x_0) = 1$. Then f and g coincide on F, so $g \in G \cap B(f, F, \epsilon)$.

Tidying Generators

Theorem

Let X be any space, and give $C(X)$ either the compact-open topology or the pointwise topology. Then the following are equivalent:

(1) X has a σ-compact $(0,1)$-generator,
(2) X has a σ-compact $(0,\neq 0)$-generator,
(3) X has a σ-compact generator,
(4) X has a compact $(0,\neq 0)$-generator, and
(5) X has a compact generator.

(1) $\Rightarrow$ (4): $G = \bigcup_n G_n$ be a $(0,1)$-generator, each G_n cpt.
Set $G_n' = \text{mid}(-1/n, G_n, 1/n)$ – compact. Set $G' = \bigcup_n G_n' \cup \{\mathbf{0}\}$.

Lemma

Generator G, set $G' = \{\mathbf{1} - \min(\lambda|g - \mu\mathbf{1}|, \mathbf{1}) : g \in G, \lambda, \mu \in \mathbb{R}\}$.
Then G' is a $(0,1)$-generator, the continuous image of $\mathbb{R}^2 \times G$.

Which spaces
have a compact generator:

1. In $C_k(X)$?
2. In $C_p(X)$?
3. With special properties?

Compact Generators - Compact-Open

Proposition

Every metrizable space X has a compact generator in $C_k(X)$.

Proposition

Let X be a k-space with a σ-compact generator in $C_k(X)$. Then X is metrizable.

Remove restriction to k-space?

Metrizable $\Rightarrow$ Compact Generator in Cpt-Open

Proposition

Every metrizable space X has a compact generator in $C_k(X)$.

Let (X, d) be a metric space with d bounded by 1.

Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be a basis for X with each $\mathcal{B}_n$ locally finite.

For any $\mathcal{B}' \subseteq \mathcal{B}$,

define $g_{\mathcal{B}'} = \sum_{n=1}^{\infty} g_{\mathcal{B}',n}/2^n$

where $g_{\mathcal{B}',n}(x) = \sup\left(\{0\} \cup D_{\mathcal{B}',n,x}\right)$

with $D_{\mathcal{B}',n,x} = \{d(x, X \setminus B) : B \in \mathcal{B}' \cap \mathcal{B}_n\}$.

Define $\Gamma : \{0,1\}^{\mathcal{B}} \to C_k(X, [0,1])$ by $\Gamma(\chi_{\mathcal{B}'}) = g_{\mathcal{B}'}$.

Let G be the image of $\{0,1\}^{\mathcal{B}}$ under Γ.

Γ is continuous, G a compact generator.

Compact Generator in Cpt-Open $\Rightarrow$ Metrizable

Proposition

*Let X be a k-space with a compact generator in $C_k(X)$.
Then X is metrizable.*

Let G be a compact $(0, \neq 0)$-generator in $C_k(X)$.

Set $W_n = \{(x, x') : \text{if } g \in G \text{ and } |g(x)| \geq 1/n \text{ then } |g(x')| > 0\}$.

Then $(W_n)_n$ is the basis for a compatible local uniformity.
So X metrizable.

As X is a k-space and G compact, the
evaluation map $e : G \times X \to \mathbb{R}$, $e(g, x) = g(x)$ is continuous.

Compact Generators - Pointwise Topology

Proposition

Space X has a compact generator in $C_p(X)$ if and only if X is Eberlein-Grothendieck.

A space is *Eberlein-Grothendieck* (EG) if it embeds in a $C_p(K)$ where K is compact.

(Arhangelskii) Internal characterization of EG spaces.

Eberlein-Grothendieck spaces have all finite powers countably tight, and *monolithic* – that is, $nw(\overline{A}) \leq |A|$ for every subspace A, and in particular, every separable subspace has a countable network.

Compact Eberlein-Grothendieck Spaces

Fact: Compact Eberlein-Grothendieck space

$\equiv$ Eberlein compact.

Eberlein compacta are *really* well understood.

Theorem

Let X be compact. Then the following are equivalent:

(1) X is Eberlein compact,

(2) X has a σ-point finite T_0-separating family of cozero sets,

(3) X has a separator homeomorphic to some $\alpha(D(\kappa))$,

(4) X has a generator homeomorphic to a continuous image of some $\alpha\left(\bigoplus_n \alpha(D(\kappa))^n\right)$.

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Theorem

Let X be compact. Then the following are equivalent:

(1) X is Eberlein compact,

(2) X has a σ-point finite almost subbase (Dimov),

(3) X has a separator homeomorphic to some $\alpha(D(\kappa))$,

(4) X has a generator homeomorphic to a continuous image of some $\alpha\left(\bigoplus_n \alpha(D(\kappa))^n\right)$.

Which spaces have a:

1. generator homeomorphic to $\alpha(D(\kappa))$?
2. generator homeomorphic to a continuous image of some $\alpha\left(\bigoplus_n \alpha(D(\kappa))^n\right)$?
3. σ-point finite almost subbase?

In particular – are they the Eberlein-Grothendieck spaces?

Very Simple Compact Generators

Theorem

A space X has a

compact $(0, \neq 0)$-generator homeomorphic to some $\alpha(D(\kappa))$

if and only if

it has a σ-point-finite base of cozero sets.

Simple Compact Generators

Theorem

Let X be a space. Then the following are equivalent:

(1) X embeds in some $C_p(\alpha(D(\kappa)))$,

(2) X has a generator that is a continuous image of some $\alpha\left(\bigoplus_n \alpha(D(\kappa))^n\right)$, and

(3) X has a σ-point-finite almost subbase.

Simple Compact Generators

Theorem

Let X be a space. Then the following are equivalent:

(1) X embeds in some $C_p(\alpha(D(\kappa)))$,
(2) X has a generator that is a continuous image of some $\alpha\left(\bigoplus_n \alpha(D(\kappa))^n\right)$, and
(3) X has a σ-point-finite almost subbase.

Let's say X is *K-Eberlein Grothendieck* if it embeds in $C_p(K)$.

Which spaces are, say, I-EG?

Reference

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