Cardinal inequalities with Shanin number and π-character

Ivan S. Gotchev
Central Connecticut State University

Joint work with Vladimir Tkachuk

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Hajnal–Juhász' and Arhangel′skiĭ's inequalities

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Theorem: [Arhangel′skiĭ's, 1969] If X is a Hausdorff space, then

$$|X| \leq 2^{\chi(X)L(X)},$$

where $L(X)$ is the Lindelöf degree of X.

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$$d(X)^{\chi(X)} \leq |X|^{\chi(X)} \leq (2^{\chi(X)c(X)})^{\chi(X)} = 2^{\chi(X)c(X)}$$

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The θ-density of a space X is $d_\theta(X) = \min\{|A| : A \subset X, \mathrm{cl}_\theta(A) = X\}$.

Bella and Cammaroto's inequality

Theorem 2: [Bella and Cammaroto, 1988] If X is a Hausdorff space, then

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Since $d_\theta(X) \leq d(X)$ for every space X, Bella and Cammaroto's inequality is a formal generalization of Pospišil's inequality.

More definitions

For every Urysohn space X the θ^2-*pseudocharacter*, denoted by $\psi_{\theta^2}(X)$, is defined to be the smallest infinite cardinal κ such that for each $x \in X$, there is a collection $\mathcal{V}_x$ of open neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and $\bigcap\{\mathrm{cl}_\theta(\overline{V}) : V \in \mathcal{V}_x\} = \{x\}$.

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We note that when $U \subset X$ is open, then $\mathrm{cl}(U) = \mathrm{cl}_\theta(U)$. Therefore, $\mathrm{cl}_\theta(\mathrm{cl}(V)) = \mathrm{cl}_\theta(\mathrm{cl}_\theta(V))$. This explains the notation $\psi_{\theta^2}(X)$ and also shows that we can use the notation $\psi_\theta(X)$ instead of $\psi_c(X)$.

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It follows from the definitions that $\psi(x) \leq \psi_c(X) \leq \psi_{\theta^2}(X) \leq \chi(X)$ for every Urysohn space X and that for regular spaces we have $\psi(x) = \psi_c(X) = \psi_{\theta^2}(X)$.

A new generalization of Pospišil's inequality

Theorem 3: [G–T, 2022] If X is a Urysohn space, then

$$|X| \leq d_\theta(X)^{\pi\chi(X)\psi_{\theta^2}(X)},$$

where $\pi\chi(X)$ is the π-character of X.

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Therefore Bella and Cammaroto's inequality is equivalent to Pospišil's inequality for Urysohn spaces.

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Therefore the inequality in Theorem 3 is not better than Willard and Dissanayake's inequality but it was useful to show that $d(X)^{\chi(X)} = d_\theta(X)^{\chi(X)}$ for every Urysohn space X.

Possible improvements of Pospišil's or Willard and Dissanayake's inequality

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Remark 1: The space $X = \beta\omega$ shows that the inequality $|X| \leq d(X)^{\pi\chi(X)}$ can fail even for compact Hausdorff spaces.

Remark 2: Fleissner gave in 1978 a very non-trivial consistent example of a space X such that $|X| > c(X)^{\chi(X)}$, so it is not possible, at least in ZFC, to replace the density with the Souslin number even in Pospišil's inequality.

Shanin's number

Therefore it is natural to ask if $d(X)$ could be replaced by the Shanin's number $sh(X)$ in Pospišil's or in Willard and Dissanayake's inequality because $c(X) \leq sh(X) \leq d(X)$ for any space X.

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Remark: $sh(X) = \min\{\kappa : \kappa^+ \text{ is a caliber of } X\}$.

Recall that a regular cardinal κ is a caliber of a space X if for any family $\mathcal{U}$ of non-empty open subsets of X such that $|\mathcal{U}| = \kappa$, there exists a family $\mathcal{V} \in [\mathcal{U}]^\kappa$ such that $\bigcap \mathcal{V} \neq \emptyset$.

Inequalities with Shanin's number

Theorem 5: [G–T, 2022] If either $\max\{\pi\chi(X), t(X)\} \geq sh(X)$ or $2^{sh(X)} = sh(X)^+$ for a regular Hausdorff space X, then $d(X) \leq sh(X)^{\pi\chi(X)\cdot t(X)}$.

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Corollary: Under GCH, if X is a regular Hausdorff space, then we have the inequality $d(X) \leq sh(X)^{\pi\chi(X)\cdot t(X)}$.

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Remark: Fleissner's example mentioned before shows that there is a model of ZFC in which GCH holds and $d(X) > c(X)^{\chi(X)}$. Therefore, consistently, we cannot replace in the above inequality $sh(X)$ with $c(X)$.

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Theorem 6: [G–T, 2022] If either $\max\{\pi\chi(X), \psi_c(X)\} \geq sh(X)$ or $2^{sh(X)} = sh(X)^+$ for a Hausdorff space X, then $|X| \leq sh(X)^{\pi\chi(X)\cdot\psi_c(X)}$.

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Inequalities with the π-weight

Observation: If X is an infinite Hausdorff space, then $d(X)^{\pi\chi(X)\cdot\psi_c(X)} = \pi w(X)^{\pi\chi(X)\cdot\psi_c(X)}$.

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Open questions

Question 1: Assume that X is a regular Hausdorff space such that $t(X) = \psi(X) = \omega$ and ω_1 is a caliber of X. Does there exist any bound on the cardinality of X? For example, is it true that $|X| \leq 2^{\mathfrak{c}}$?

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Question 3: Assume that X is a regular σ-closed, discrete Hausdorff space of countable tightness such that ω_1 is a caliber of X. Does there exist any bound on the cardinality of X? For example, is it true that $|X| \leq 2^{\mathfrak{c}}$?

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The End

THANK YOU!