Pure point diffraction and entropy beyond the euclidean space

Till Hauser

MPI Bonn

July 21, 2022

Introduction

Naturhistorisches Museum Wien (wikipedia.org)

(www.nhm-wien.ac.at)

Introduction

0. Introduction

0. Introduction
1. Dyadic numbers

Introduction

Introduction

0. Introduction
1. Dyadic numbers
2. Delone sets
   2.1 Pure point diffraction
   2.2 Patch counting entropy
3. Euclidean case.
4. Dyadic case (Main result).

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Remark (Basic Idea of dyadic numbers $\mathbb{Q}_2$)

For $(x_k)_{k=-m}^n$ in $\{0,1\}$:

$$x_n \dots x_1 . x_0 x_{(-1)} \dots x_{(-m)} := \sum_{k=-m}^{n} x_k 2^k.$$

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Remark (Basic Idea of dyadic numbers $\mathbb{Q}_2$)

For $(x_k)_{k=-m}^{n}$ in $\{0,1\}$:

$$x_n \dots x_1 . x_0 x_{(-1)} \dots x_{(-m)} := \sum_{k=-m}^{n} x_k 2^k.$$

$\mathbb{Z}[1/2] :=$ *set of rational numbers with finite binary expansion as above.*

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Remark (Basic Idea of dyadic numbers $\mathbb{Q}_2$)

For $(x_k)_{k=-m}^{n}$ in $\{0,1\}$:

$$x_n \dots x_1 . x_0 x_{(-1)} \dots x_{(-m)} := \sum_{k=-m}^{n} x_k 2^k.$$

$\mathbb{Z}[1/2]$:= *set of rational numbers with finite binary expansion as above.*

$\mathbb{R}$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to standard metric.*

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Remark (Basic Idea of dyadic numbers $\mathbb{Q}_2$)

For $(x_k)_{k=-m}^{n}$ in $\{0,1\}$:

$$x_n \dots x_1 . x_0 x_{(-1)} \dots x_{(-m)} := \sum_{k=-m}^{n} x_k 2^k.$$

$\mathbb{Z}[1/2] :=$ *set of rational numbers with finite binary expansion as above.*

$\mathbb{R}$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to standard metric.*

$\mathbb{Q}_2$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to 'mirrored' metric.*

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Remark (Basic Idea of dyadic numbers $\mathbb{Q}_2$)

For $(x_k)_{k=-m}^n$ in $\{0,1\}$:

$$x_n \dots x_1 . x_0 x_{(-1)} \dots x_{(-m)} := \sum_{k=-m}^{n} x_k 2^k.$$

$\mathbb{Z}[1/2] :=$ *set of rational numbers with finite binary expansion as above.*

$\mathbb{R}$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to standard metric.*

$\mathbb{Q}_2$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to 'mirrored' metric.*

$\mathbb{R}, \mathbb{Q}_2$ *field extensions of* $\mathbb{Q}$.

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Remark (Basic Idea of dyadic numbers $\mathbb{Q}_2$)

For $(x_k)_{k=-m}^{n}$ in $\{0,1\}$:

$$x_n \dots x_1 . x_0 x_{(-1)} \dots x_{(-m)} := \sum_{k=-m}^{n} x_k 2^k.$$

$\mathbb{Z}[1/2] :=$ *set of rational numbers with finite binary expansion as above.*

$\mathbb{R}$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to standard metric.*

$\mathbb{Q}_2$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to 'mirrored' metric.*

$\mathbb{R}, \mathbb{Q}_2$ *field extensions of* $\mathbb{Q}$.

$0.010101010101\dots$

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Remark (Basic Idea of dyadic numbers $\mathbb{Q}_2$)

For $(x_k)_{k=-m}^n$ in $\{0,1\}$:

$$x_n \dots x_1 . x_0 x_{(-1)} \dots x_{(-m)} := \sum_{k=-m}^{n} x_k 2^k.$$

$\mathbb{Z}[1/2] :=$ *set of rational numbers with finite binary expansion as above.*

$\mathbb{R}$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to standard metric.*

$\mathbb{Q}_2$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to 'mirrored' metric.*

$\mathbb{R}, \mathbb{Q}_2$ *field extensions of* $\mathbb{Q}$.

$0.010101010101\dots$

$\dots 010101011.0$

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Remark (Basic Idea of dyadic numbers $\mathbb{Q}_2$)

For $(x_k)_{k=-m}^n$ in $\{0,1\}$:

$$x_n \dots x_1 . x_0 x_{(-1)} \dots x_{(-m)} := \sum_{k=-m}^{n} x_k 2^k.$$

$\mathbb{Z}[1/2] :=$ *set of rational numbers with finite binary expansion as above.*

$\mathbb{R}$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to standard metric.*

$\mathbb{Q}_2$ *'completion' of* $\mathbb{Z}[1/2]$ *with respect to 'mirrored' metric.*

$\mathbb{R}, \mathbb{Q}_2$ *field extensions of* $\mathbb{Q}$.

$$0.010101010101\dots \qquad = \frac{1}{3} = \qquad \dots 010101011.0$$

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Definition (Delone set)

Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

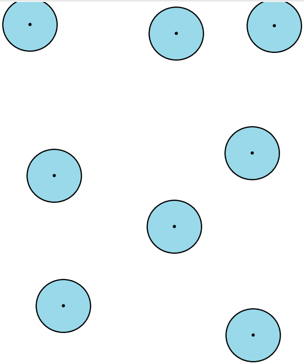
$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Definition (Delone set)

$\omega \subseteq G$ *uniformly discrete*, whenever there is an open neighbourhood $V \subseteq G$ such that $\{V + x; x \in \omega\}$ is a disjoint family.



Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

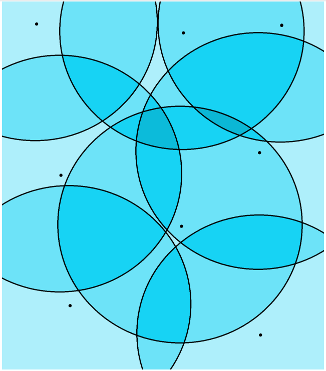
Example

Additive group of 2-adic numbers $\mathbb{Q}_2$.

Definition (Delone set)

$\omega \subseteq G$ *uniformly discrete*, whenever there is an open neighbourhood $V \subseteq G$ such that $\{V + x;\ x \in \omega\}$ is a disjoint family.

$\omega \subseteq G$ *relatively dense*, whenever there is a compact subset $K \subseteq G$ such that $\bigcup_{x\in\omega}(K + x) = G$.



Delone sets

Setting

G σ-compact locally compact Abelian group (σ-cpt. LCA group).

Example

$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T}^d$, ...

Example

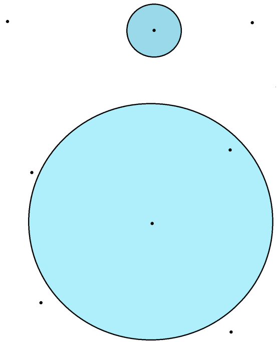
Additive group of 2-adic numbers $\mathbb{Q}_2$.

Definition (Delone set)

$\omega \subseteq G$ *uniformly discrete*, whenever there is an open neighbourhood $V \subseteq G$ such that $\{V + x;\, x \in \omega\}$ is a disjoint family.

$\omega \subseteq G$ *relatively dense*, whenever there is a compact subset $K \subseteq G$ such that $\bigcup_{x \in \omega}(K + x) = G$.

$\omega \subseteq G$ *Delone*, whenever ω uniformly discrete and relatively dense.



Patches of a Delone set

Definition (Patches)

Let $\omega \subseteq G$ be a Delone set. For compact subsets $A \subseteq G$ we define the *set of A-patches* as

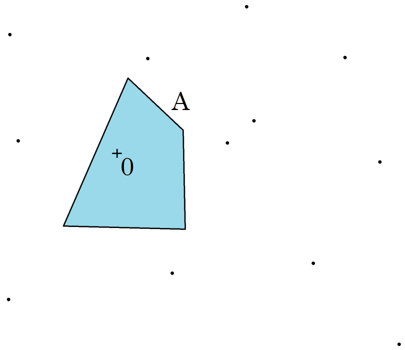
$$\mathrm{Pat}_\omega(A) := \{(\omega - g) \cap A;\ g \in \omega\}.$$

Patches of a Delone set

Definition (Patches)

Let $\omega \subseteq G$ be a Delone set. For compact subsets $A \subseteq G$ we define the *set of A-patches* as

$$\mathrm{Pat}_\omega(A) := \{(\omega - g) \cap A;\ g \in \omega\}.$$

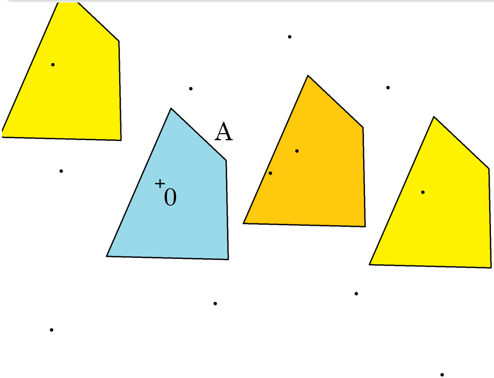


Patches of a Delone set

Definition (Patches)

Let $\omega \subseteq G$ be a Delone set. For compact subsets $A \subseteq G$ we define the *set of A-patches* as

$$\mathrm{Pat}_{\omega}(A) := \{(\omega - g) \cap A;\ g \in \omega\}.$$

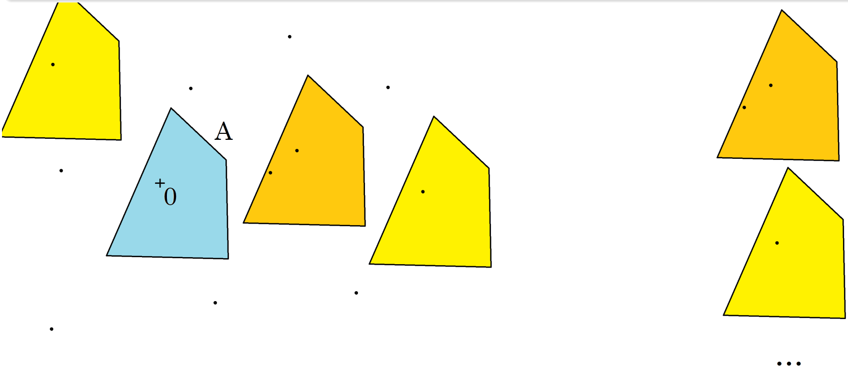


Patches of a Delone set

Definition (Patches)

Let $\omega \subseteq G$ be a Delone set. For compact subsets $A \subseteq G$ we define the *set of A-patches* as

$$\mathrm{Pat}_\omega(A) := \{(\omega - g) \cap A;\ g \in \omega\}.$$



Finite local complexity

Definition (Patches)

$\mathrm{Pat}_\omega(A) := \{(\omega - g) \cap A;\ g \in \omega\}$

Finite local complexity

Definition (Patches)

$\mathrm{Pat}_\omega(A) := \{(\omega - g) \cap A;\ g \in \omega\}$

Definition (FLC)

$\omega \subseteq G$ is called *FLC* (of finite local complexity), if $\mathrm{Pat}_\omega(A)$ is finite for all compact subsets $A \subseteq G$.

Finite local complexity

Definition (Patches)

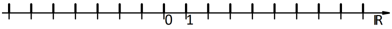
$\mathrm{Pat}_\omega(A) := \{(\omega - g) \cap A;\ g \in \omega\}$

Definition (FLC)

$\omega \subseteq G$ is called *FLC* (of finite local complexity), if $\mathrm{Pat}_\omega(A)$ is finite for all compact subsets $A \subseteq G$.

Example

$\mathbb{Z} \subseteq \mathbb{R}$ is a FLC Delone set.



Finite local complexity

Definition (Patches)

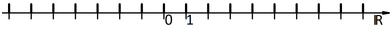
$\mathrm{Pat}_\omega(A) := \{(\omega - g) \cap A;\ g \in \omega\}$

Definition (FLC)

$\omega \subseteq G$ is called *FLC* (of finite local complexity), if $\mathrm{Pat}_\omega(A)$ is finite for all compact subsets $A \subseteq G$.

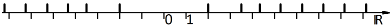
Example

$\mathbb{Z} \subseteq \mathbb{R}$ is a FLC Delone set.



Example

$\{n + 1/n;\ n \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{R}$ is a Delone set but not FLC.



Finite local complexity

Definition (Patches)

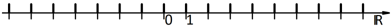
$\mathrm{Pat}_\omega(A) := \{(\omega - g) \cap A;\ g \in \omega\}$

Definition (FLC)

$\omega \subseteq G$ is called *FLC* (of finite local complexity), if $\mathrm{Pat}_\omega(A)$ is finite for all compact subsets $A \subseteq G$.

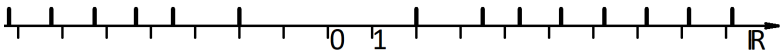
Example

$\mathbb{Z} \subseteq \mathbb{R}$ is a FLC Delone set.



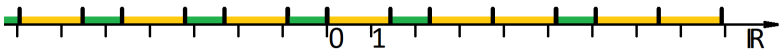
Example

$\{n + 1/n;\ n \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{R}$ is a Delone set but not FLC.



Example

FLC Delone set:



Definition: pure point diffraction and entropy

Setting

$G = \mathbb{Q}_2$, or $G = \mathbb{R}^d$.

Definition: pure point diffraction and entropy

Setting

$G = \mathbb{Q}_2$, or $G = \mathbb{R}^d$.

Definition (Pure point diffraction)

A FLC Delone set ω is called *pure point diffractive* (PPD), whenever

(i) $\frac{1}{\vartheta(B_n)} \sum_{g \in (B_n \cap \omega) - (B_n \cap \omega)} \delta_g$ converges in the weak*-topology (to γ).

(ii) The Fourier transform $\hat{\gamma}$ of γ is a pure point measure.

Definition: pure point diffraction and entropy

Setting

$G = \mathbb{Q}_2$, or $G = \mathbb{R}^d$.

Definition (Pure point diffraction)

A FLC Delone set ω is called *pure point diffractive* (PPD), whenever

(i) $\frac{1}{\vartheta(B_n)} \sum_{g \in (B_n \cap \omega) - (B_n \cap \omega)} \delta_g$ converges in the weak*-topology (to γ).

(ii) The Fourier transform $\hat{\gamma}$ of γ is a pure point measure.

Definition (Patch counting entropy)

We define the *patch counting entropy* of a FLC Delone set $\omega \subseteq G$ as

$$h_{pat}(\omega) := \limsup_{n \to \infty} \frac{\log |\operatorname{Pat}_\omega(B_n)|}{\vartheta(B_n)}.$$

Main result

Theorem (J. Lagarias., T.H.)

For any FLC Delone set $\omega \subseteq \mathbb{R}^d$

$$\left(\frac{\log|\operatorname{Pat}_\omega(B_n)|}{\vartheta(B_n)}\right)_{n\in\mathbb{N}}$$

converges to $\mathrm{h}_{pat}(\omega) < \infty$.

Main result

Theorem (J. Lagarias., T.H.)

For any FLC Delone set $\omega \subseteq \mathbb{R}^d$

$$\left(\frac{\log|\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)}\right)_{n\in\mathbb{N}}$$

converges to $\mathrm{h}_{pat}(\omega) < \infty$.

Theorem (M. Baake, D. Lenz, C. Richard)

For any PPD FLC Delone set $\omega \subseteq \mathbb{R}^d$

$$\left(\frac{\log|\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)}\right)_{n\in\mathbb{N}}$$

converges to 0.

Main result

Theorem (J. Lagarias., T.H.)

For any FLC Delone set $\omega \subseteq \mathbb{R}^d$

$$\left(\frac{\log|\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)}\right)_{n\in\mathbb{N}}$$

converges to $\mathrm{h}_{pat}(\omega) < \infty$.

Theorem (M. Baake, D. Lenz, C. Richard)

For any PPD FLC Delone set $\omega \subseteq \mathbb{R}^d$

$$\left(\frac{\log|\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)}\right)_{n\in\mathbb{N}}$$

converges to 0.

Theorem (H. 2022)

For $0 \leq s \leq r \leq \infty$ *there exists a PPD FLC Delone set* $\omega \subseteq \mathbb{Q}_2$ *such that*

(i) $\mathrm{h}_{pat}(\omega) = r$,

(i) $\liminf_{n\to\infty} \frac{\log|\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)} = s$.

Remark

(i) *non-existence of the limit.*

Main result

Theorem (J. Lagarias., T.H.)

For any FLC Delone set $\omega \subseteq \mathbb{R}^d$

$$\left(\frac{\log|\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)}\right)_{n\in\mathbb{N}}$$

converges to $\mathrm{h}_{pat}(\omega) < \infty$.

Theorem (M. Baake, D. Lenz, C. Richard)

For any PPD FLC Delone set $\omega \subseteq \mathbb{R}^d$

$$\left(\frac{\log|\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)}\right)_{n\in\mathbb{N}}$$

converges to 0.

Theorem (H. 2022)

For $0 \leq s \leq r \leq \infty$ *there exists a PPD FLC Delone set* $\omega \subseteq \mathbb{Q}_2$ *such that*

(i) $\mathrm{h}_{pat}(\omega) = r$,

(i) $\liminf_{n\to\infty} \frac{\log|\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)} = s$.

Remark

(i) *non-existence of the limit.*

(ii) $\mathrm{h}_{pat}(\omega) = \infty$ *possible.*

Main result

Theorem (J. Lagarias., T.H.)

For any FLC Delone set $\omega \subseteq \mathbb{R}^d$

$$\left(\frac{\log |\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)}\right)_{n\in\mathbb{N}}$$

converges to $\mathrm{h}_{pat}(\omega) < \infty$.

Theorem (M. Baake, D. Lenz, C. Richard)

For any PPD FLC Delone set $\omega \subseteq \mathbb{R}^d$

$$\left(\frac{\log |\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)}\right)_{n\in\mathbb{N}}$$

converges to 0.

Theorem (H. 2022)

For $0 \leq s \leq r \leq \infty$ *there exists a PPD FLC Delone set* $\omega \subseteq \mathbb{Q}_2$ *such that*

(i) $\mathrm{h}_{pat}(\omega) = r$,

(i) $\liminf_{n\to\infty} \frac{\log |\mathrm{Pat}_\omega(B_n)|}{\vartheta(B_n)} = s$.

Remark

(i) *non-existence of the limit.*

(ii) $\mathrm{h}_{pat}(\omega) = \infty$ *possible.*

(iii) *'regular model sets'* $\omega \subseteq \mathbb{Q}_2$ *satisfy* $\mathrm{h}_{pat}(\omega) = 0$ *[C. Huck, C. Richard].*

Better notion of patch counting entropy

Definition

ω Delone set. $B \subseteq G$ compact, V open neighbourhood of 0.
$F \subseteq \omega$ is called a *B-patch representation at scale V for ω*, whenever

$$\forall g \in G \exists f \in F \colon (\omega - f) \cap B \subseteq (\omega - g) + V \text{ and v.v..}$$

$\mathrm{pat}_\omega(B, V) :=$ minimal cardinality of such a representation.

Better notion of patch counting entropy

Definition

ω Delone set. $B \subseteq G$ compact, V open neighbourhood of 0.
$F \subseteq \omega$ is called a *B-patch representation at scale V for ω*, whenever

$$\forall g \in G \exists f \in F \colon (\omega - f) \cap B \subseteq (\omega - g) + V \text{ and v.v..}$$

$\mathrm{pat}_\omega(B, V) :=$ minimal cardinality of such a representation.

Definition

$h'_{pat}(\omega) := \sup_V \limsup_{n\to\infty} \frac{\log \mathrm{pat}_\omega(B_n, V)}{\vartheta(B_n)}$.

Better notion of patch counting entropy

Definition

ω Delone set. $B \subseteq G$ compact, V open neighbourhood of 0.
$F \subseteq \omega$ is called a *B-patch representation at scale V for ω*, whenever

$$\forall g \in G \exists f \in F \colon (\omega - f) \cap B \subseteq (\omega - g) + V \text{ and v.v..}$$

$\mathrm{pat}_\omega(B, V) :=$ minimal cardinality of such a representation.

Definition

$h'_{pat}(\omega) := \sup_V \limsup_{n\to\infty} \frac{\log \mathrm{pat}_\omega(B_n, V)}{\vartheta(B_n)}$.

Remark

All FLC Delone sets in $\mathbb{R}^d$ satisfy $h_{pat}(\omega) = h'_{pat}(\omega)$.

Better notion of patch counting entropy

Definition

ω Delone set. $B \subseteq G$ compact, V open neighbourhood of 0.
$F \subseteq \omega$ is called a *B-patch representation at scale V for ω* , whenever

$$\forall g \in G \exists f \in F \colon (\omega - f) \cap B \subseteq (\omega - g) + V \text{ and v.v..}$$

$\mathrm{pat}_\omega(B, V) :=$ minimal cardinality of such a representation.

Definition

$h'_{pat}(\omega) := \sup_V \limsup_{n\to\infty} \frac{\log \mathrm{pat}_\omega(B_n, V)}{\vartheta(B_n)}$.

Remark

All FLC Delone sets in $\mathbb{R}^d$ satisfy $h_{pat}(\omega) = h'_{pat}(\omega)$.

Theorem (T.H.)

All PPD FLC Delone sets in $\mathbb{Q}_2$ satisfy $h'_{pat}(\omega) = 0$.