The hyperspace of noncut subcontinua of a hairy dendrite

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Joint work with Jorge Vega

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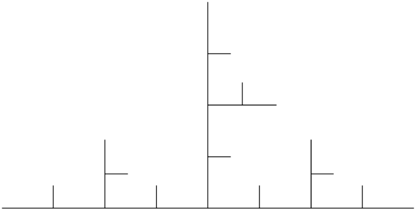
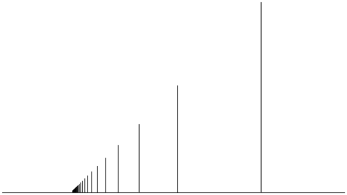
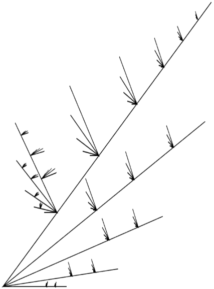
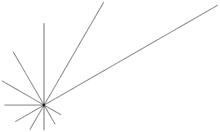
Dendrites

A **dendrite** is a locally connected metric continuum that does not contain simple closed curves.









Menger-Urysohn order

Let X be a dendrite and $p \in X$. The **Menger-Urysohn order** of p in X is the number of components of $X \setminus \{p\}$ and is denoted by $\mathrm{ord}(p, X)$.

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E(X) &= \{p \in X : \mathrm{ord}(p, X) = 1\} && \text{endpoints} \\
O(X) &= \{p \in X : \mathrm{ord}(p, X) = 2\} && \text{ordinary points} \\
R(X) &= \{p \in X : \mathrm{ord}(p, X) \geq 3\} && \text{ramification points}
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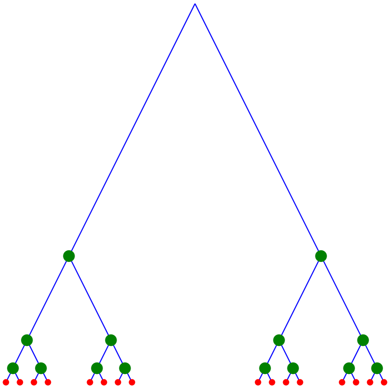
Important: For dendrites, $\mathrm{ord}(p, X)$ can be any finite number or ω.

Example: points by their order

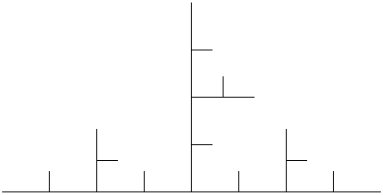


“Hairy” dendrites

Lemma (J. Chatatonik, W. Charatonik and J. Prajs, 1994)

For a dendrite X, the following are equivalent.

1. $E(X)$ *is dense,*
2. $R(X)$ *is dense, and*
3. *if* α *is an arc in* X*, then* $\alpha \cap R(X)$ *is dense.*



Non-cut subcontinua

Let X be a metric continuum; then

$$\begin{aligned} 2^X &= \{A \subset X : A \text{ is closed and nonempty}\}, \\ C(X) &= \{A \in 2^X : A \text{ is a continuum}\}. \end{aligned}$$

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$$NC^*(X) = \{A \in C(X) \colon X \setminus A \text{ is connected}\}.$$

Total disconnected and ... ?

Theorem (Jorge Martinez-Montejano, Verónica Martinez-de-la-Vega and Jorge Vega)

If X is a dendrite where $R(X)$ is dense, then $NC^*(X)$ is totally disconnected.

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Theorem (HG and Vega)

If X is a dendrite where $R(X)$ is dense, then $NC^*(X) \approx \mathbb{R} \setminus \mathbb{Q}$.

Characterization of $\mathbb{R} \setminus \mathbb{Q}$

Theorem (Alexandroff and Urysohn, 1928)

For a separable metrizable space X the following are equivalent:

- $X \approx \mathbb{R} \setminus \mathbb{Q}$, *and*
- X *is Polish, zero dimensional and nowhere locally compact.*

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$$NC^*(X) = C(X) \setminus \mathcal{S}(X)$$

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Elements of $NC^*(X)$ when X is a dendrite

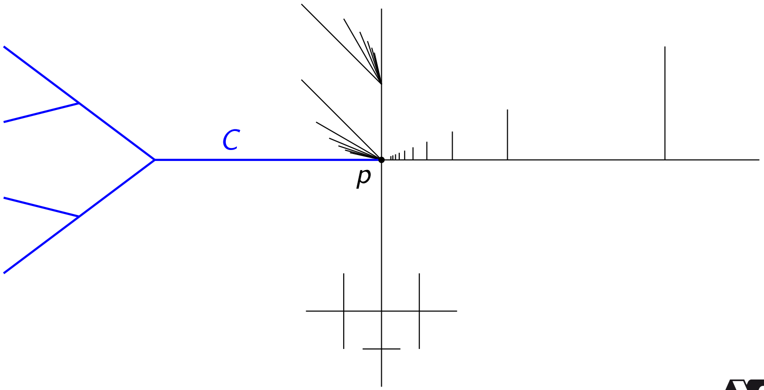
Theorem (Martinez-Montejano, Martinez-de-la-Vega and Vega)

Let X be a dendrite and let $A \in C(X)$. Then $A \in NC^*(X)$ if and only if one of the following holds:

- $A = X$,
- $A = \{e\}$ *for some* $e \in E(X)$, *or*
- $A = X \setminus C$, *where C is a component of* $X \setminus \{p\}$ *with* $p \in X \setminus E(X)$.

Example of $A \in NC^*(X)$.

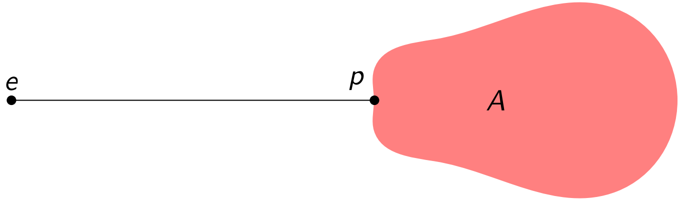
$$A = X \setminus C$$



Notice: $\mathrm{bd}_X(A) = \{p\}$.

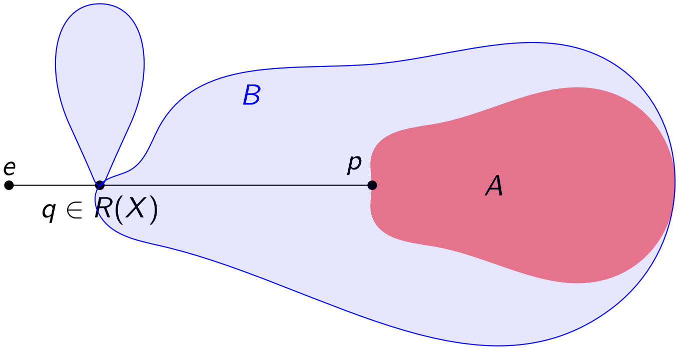
Some closed discrete sets

Let $A \in NC^*(X)$.



Some closed discrete sets

Let $A \in NC^*(X)$. Choose $B \in NC^*(X)$ close to A such that $q \in R(X)$.



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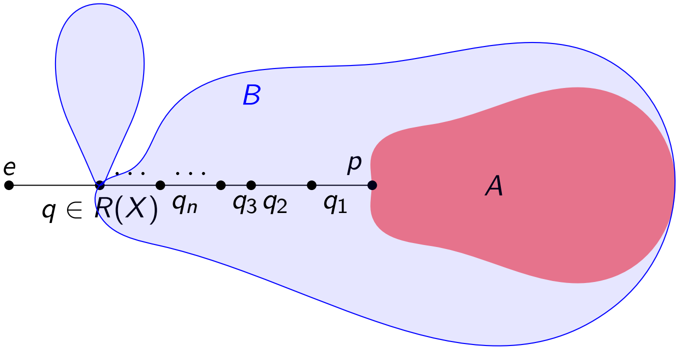

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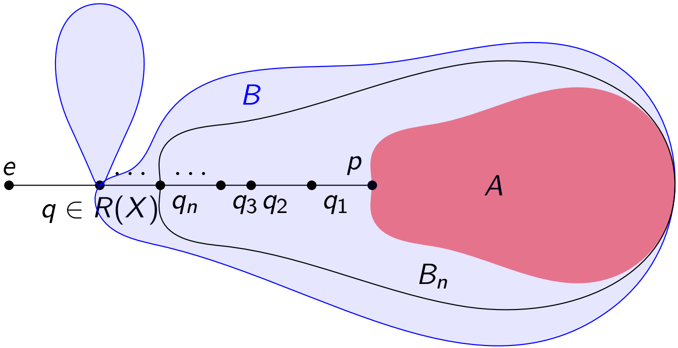

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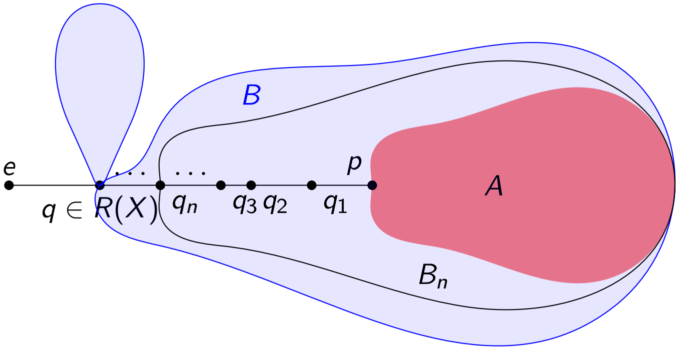

Then $\{B_n \colon n \in \mathbb{N}\}$ is closed and discrete in $NC^*(X)$.

Clopen sets in $NC^*(X)$

Let $q, r \in [ab \setminus \{a, b\}] \cap R(X)$.

$$\mathcal{B}(q, r) = \{B_x : x \in qr \setminus \{q\}\}$$

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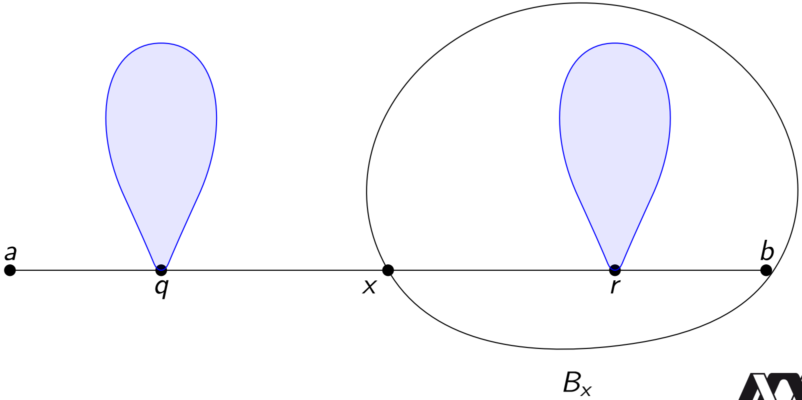

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Open questions

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In general, what space is $NC^*(X)$ *when* X *is a dendroid?*

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Is there a dendroid X *such that* $NC^*(X)$ *is totally disconnected but not zero dimensional?*

Question

If X *is the Mohler-Nikiel universal smooth dendroid, is* $NC^*(X)$ *totally disconnected and not zero-dimensional?*

Thank you

Preprint available at:
https://arxiv.org/abs/2108.06020

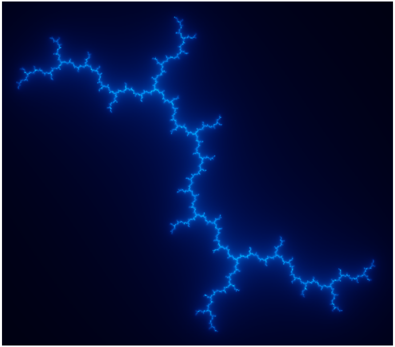

Figure: The Julia set of $z \mapsto z^2 + i$ is homeomorphic to D_3.
https://sciencedemos.org.uk/julia.php

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