1 On the Hilbert Cube

The (countable) Hilbert cube is defined as the (countable) cartesian product of the (euclidean) unit interval $[0,1]$ with itself:

$$\mathcal{H} = [0,1]^{\mathbb{N}} = \{(x_n)_{n\in\mathbb{N}} : x_n \in [0,1] \text{ for all } n \in \mathbb{N}\}.$$

We equip $\mathcal{H}$ is product topology, that is the topology determined by the basis

$$\mathcal{B}_0 := \{B \subset \mathcal{H} : \exists N \in \mathbb{N}\ \exists \text{ open } U_j \text{ for } j \le N \text{ such that } (x_n) \in B \Leftrightarrow x_j \in U_j \text{ for } j \le N\}.$$

Hence there are no restriction for the coordinates x_j with $j > N$. Since open sets in $[0,1]$ are unions of open intervals (in the relative topology, so e.g. $[0,1/2)$ is open too), a simpler basis is

$$\mathcal{B} := \{B \subset \mathcal{H} : \exists N \in \mathbb{N}\ \exists \text{ open intervals } I_j \text{ for } j \le N \text{ such that } (x_n) \in B \Leftrightarrow x_j \in I_j \text{ for } j \le N\}.$$

A subbasis for this $\mathcal{B}'$ is the following:

$$\begin{aligned} \mathcal{S} &= \{S \subset \mathcal{H} : \exists N \in \mathbb{N}\ \exists a \in [0,1) \text{ such that } (x_n) \in B \Leftrightarrow x_N \in (a,1]\} \\ &\quad \cup \{S \subset \mathcal{H} : \exists N \in \mathbb{N}\ \exists b \in (0,1] \text{ such that } (x_n) \in B \Leftrightarrow x_N \in [0,b)\}. \end{aligned}$$

The Hilbert cube with product topology is metric. A valid metric, that generated this topology, is

$$d_{\text{prod}}(x,y) = \sum_{n\in\mathbb{N}} 2^{-n}|x_n - y_n|.$$

To show that this metric indeed produces the product topology, first observe that open balls $B_\varepsilon(x)$ in the metric d_{prod} are open set. Indeed, given $\varepsilon > 0$, there is N such that $2^N\varepsilon > 2$. For $y \in B_\varepsilon(x)$, $d_{\text{prod}}(x,y) < \sum_{n>N} 2^{-n}$, so for $j \le N$ we can find open intervals $I_j \ni y_j$ of length $(\varepsilon - d_{\text{prod}}(x,y))/3$. Then

$$y \in B := I_0 \times I_1 \times I_2 \times \dots I_N \times [0,1]^{\mathbb{N}\setminus\{0,1,\dots,N\}} \in \mathcal{B} \quad \text{and} \quad B \subset B_\varepsilon(x).$$

Conversely, let $B = I_0 \times I_1 \times \cdots \times I_N \times [0,1]^{\mathbb{N}\setminus\{0,1,\dots,N\}} \in \mathcal{B}$, and take $x \in B$ arbitrary. Take $\varepsilon_j > 0$ for all $0 \le J \le N$ so small that the euclidean balls $B_{\varepsilon_j}(x_j) \subset I_j$, and $\varepsilon = \min\{\varepsilon_j : 0 \le j \le N\}$. For the metric d_{prod} this means that $B_\varepsilon(x) \subset B$, so every point in B is interior.

Lemma 1 *The Hilbert cube is compact.*

For the proof, we use the characterization of compact sets given by Alexander:

Theorem 2 (Theorem of Alexander) *Let (X,τ) be a topological space with a subbasis $\mathcal{S}$. Then $A \subset X$ is compact if and only if every cover of A using only sets in $\mathcal{S}$ has a finite subcover.*

Proof of Lemma 1. Let $\mathcal{U} \subset \mathcal{S}$ be an open cover of $\mathcal{H}$. Write

$$U^-_{N,a} := [0,1]^{N-1} \times (a,1] \times [0,1]^{\mathbb{N}\setminus\{0,1,\dots,N\}} \quad \text{and} \quad U^+_{N,b} := [0,1]^{N-1} \times [0,b) \times [0,1]^{\mathbb{N}\setminus\{0,1,\dots,N\}}$$

for the elements of $\mathcal{S}$. Now let

$$a_N = \inf\{a : U^-_{N,a} \in \mathcal{U}\} \quad \text{and} \quad b_N = \sup\{b : U^+_{N,a} \in \mathcal{U}\}.$$

If there is N such that $a_N < b_N$, then we can find $a_N \leq a < b \leq b_N$ such that both $U^-_{N,a}, U^+_{N,b} \in \mathcal{U}$. But then $\{U^-_{N,a}, U^+_{N,b}\}$ forms a finite subcover of $\mathcal{H}$, and we are done.

Otherwise, $x = (a_0, a_1, a_2, \dots) \notin \bigcup_{U \in \mathcal{U}} U$, contradicting that $\mathcal{U}$ is a cover. $\square$

Compactness of the Hilbert cube holds in the product topology, but **not** in the topology that comes from the sup-metric $d_\infty(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$. Indeed, let $B^\infty_\varepsilon(e_n)$ be the open ε-ball in the metric d_∞ around the n-th basic vector $e_n = (0, 0, 0, \dots, 0, 1, 0, \dots)$, with the 1 at place n. Then $\{B^\infty_{1/2}(e_n)\}_{n \in \mathbb{N}} \cup \{\mathcal{H} \setminus \overline{\cup_{n \in \mathbb{N}} B^\infty_{1/4}(e_n)}\}$ is an open cover of $\mathcal{H}$, but since all the balls $B_{1/2}(e_n)$ are pairwise disjoint, there is no finite subcover.

Note also that for example $B^\infty_{1/2}(e_1)$ is not open in the product topology, because it restricts all (infinitely many) coordinates. Indeed, if $B^\infty_{1/2}(e_1)$ was open in the product topology, then it should be possible to write it as union of elements in $\mathcal{B}$. However, every $B \in \mathcal{B}$ contains a sequence x for which $x_n = 1$ for n sufficiently large. So $B \not\subset B^\infty_{1/2}(e_1)$.