Topological Factoring of Minimal Cantor Systems

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Minimal Cantor System.

- X Cantor set, $T : X \longrightarrow X$ a homeomorphism.

$$\mathbb{Z} \times X \longrightarrow X$$
$$(n, x) \mapsto T^n x.$$

- $\forall x \in X : \quad \overline{\mathcal{O}(x)} = X.$

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$$(X, T) \ \text{Cantor minimal system}$$

- Examples: Odometer, Denjoy's, Sturmians, Substitutions systems,...
- $\mathcal{M}_T(X) = \{\mu : \ T\mu = \mu\}.$
- Topological Factoring:

$$\pi \circ T = S \circ \pi, \qquad \begin{array}{ccc} X & \xrightarrow{T} & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{S} & Y \end{array}$$

Dimension Group.

For (X, T) consider $C(X, \mathbb{Z})$ and

- the subgroup of Co-boundaries:

$$(1-T)C(X,\mathbb{Z}) = \{f \in C(X,\mathbb{Z}) : \ \exists g \in C(X,\mathbb{Z}), \ f = g - g \circ T\}.$$

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- $K^0(X, T)$ can be computed as a direct limit of finitely generated groups and positive homomorphisms.

- **[G. Elliott, '76]**

$$G = \varinjlim_i (\mathbb{Z}^{n_i}, M_i).$$

$$\mathbb{Z}^{n_1} \xrightarrow{M_1} \mathbb{Z}^{n_2} \xrightarrow{M_2} \mathbb{Z}^{n_3} \xrightarrow{M_3} \cdots \longrightarrow \mathbb{G}$$

$$G = \coprod_i \mathbb{Z}^{r(i)} / \sim \quad \text{with} \quad [g,\, i] \sim [h,\, j] \;\; \text{iff}$$

$$\exists\; k > i, j; \;\; M_i \circ M_{i+1} \circ \cdots \circ M_k(g) = M_j \circ M_{j+1} \circ \cdots \circ M_k(h).$$

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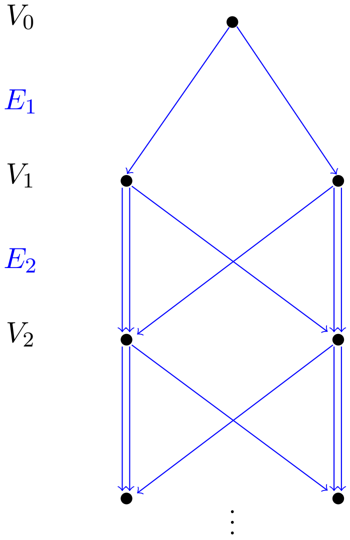
- So we have a Bratteli diagram.

$$V_{n_1} \xrightarrow{E_1} V_{n_2} \xrightarrow{E_2} V_{n_3} \xrightarrow{E_3} \cdots$$

Example.



Example:

- $$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots \longrightarrow \mathbb{G} \simeq \mathbb{Z}[\tfrac{1}{2}].$$

$$[b, m] \sim [a,\, n], m \leq n \;\Leftrightarrow\; 2^{k-m}b = 2^{k-n}a \;\Leftrightarrow\; b = \frac{a}{2^{n-m}}.$$

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- $$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \cdots \longrightarrow \mathbb{G} \simeq \mathbb{Z}[\tfrac{1}{2}].$$

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- $$\mathbb{Z}^2 \xrightarrow{\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix}} \cdots \longrightarrow \mathbb{G} \simeq \mathbb{Z} + \theta\mathbb{Z}.$$

  where $\theta = [a_1, a_2, a_3, \cdots]$.

Kakutani-Rokhlin Partitions.

Take any point $x_0 \in X$ and a clopen set C around that.

$$\forall x \in C: \ n_C : X \to \mathbb{Z}; \ n_C(x) = \inf\{n > 0 : \ T^n x \in C\}.$$

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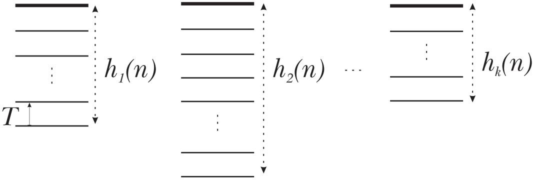

Using minimality, the above towers make a partition of (X, T).

Choosing $C' \subset C$ with $x_0 \in C'$ we get a finer partition and by continuing this procedure:

Theorem. (I. Putnam, 1989)

Let (X, T) be a minimal Cantor system and $x_0 \in X$. There exists a nested and refining sequence of K-R partitions, say $\mathcal{P} = \{\mathcal{P}_i\}_{i \geq 0}$, of X that their basis converge to $\{x_0\}$.

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For the $\mathcal{P} = \{\mathcal{P}_i\}_{i \geq 0}$, let

$$L_{\mathcal{P}} := \sup_i \#\{\text{towers in } \mathcal{P}_{\rm i}\}$$

then

$$\text{rank}_{\text{top}}(X, T) = \inf_{\mathcal{P}} \; L_{\mathcal{P}}.$$

Example.

The odometers: $\mathbb{Z}_2 \xleftarrow{\varphi_1} \mathbb{Z}_4 \xleftarrow{\varphi_2} \mathbb{Z}_8 \longleftarrow \cdots \longleftarrow \mathbb{Z}_{(2^n)}$
where

$$\mathbb{Z}_{(p_n)} = \{(x_n)_{n\geq 1} \in \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n} : \ \varphi_n(x_n+1) = x_n\}, \quad p_n | p_{n+1}.$$

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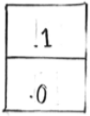
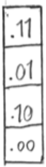
Consider $x = 0000\cdots \in C = [0]$.

$$\mathcal{P}_1 : [0] \mapsto [1]$$

and with $C_1 = [00]$,

$$\mathcal{P}_2 : [00] \mapsto [10] \mapsto [01] \mapsto [11] \mapsto [00].$$





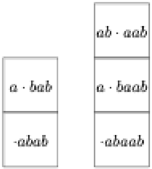
Example.

The two-sided Sturmian subshift generated by the morphism

$$\tau : \; a \mapsto ab, \quad b \mapsto a$$

and $C = [ab], x$ the unique fixed point of τ. Then

$$\mathcal{P} = \{.abab, \sigma(.abab), .abaab, \sigma(.abaab), \sigma^2(.abaab)\}$$



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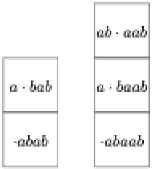
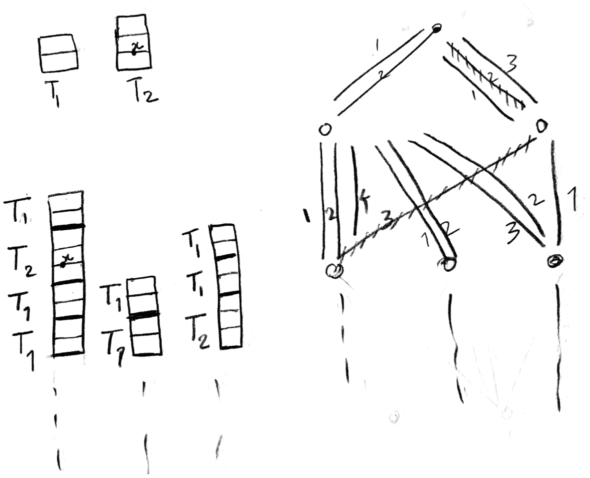


there exists a nested sequence of K-R partitions $\mathcal{P}_i, i \geq 2$ *that refine* $\mathcal{P}$ *and they all have exactly 2 towers.* So the system is of topological rank 2.

Bratteli Diagram Associated to the K-R partitions.



Going one level up on the towers $\sim$ Mapping an infinite path on its successor.

Realization of Cantor minimal systems

Theorem (Herman-Putnam-Skau, 1992)

Let (X, T) be a Cantor minimal system and fix $x_0 \in X$. Then there is a unique (up to equivalence) simple properly ordered Bratteli diagram B such that (X, T) is conjugate to (X_B, λ_B) with a conjugating map $\alpha\colon X \to X_B$ satisfying $\alpha(x) = x_{\min}$.

Remark. $K^0(X, T)$ is the direct limit of sequence coming up from the Bratteli diagram.

Algebraic Rank.

$$\text{rank}(G) = \text{Dimension of the vector space on } G \text{ over } \mathbb{Q}.$$

Example.

- for all the dometers $K^0(X,T) \subseteq \mathbb{Q}$ and so $\text{rank}(K^0(X,T)) = 1$.

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- for a sturmian system associated to an irrational rotation with rotation θ:

  $$K^0(X,T) = \mathbb{Z} + \theta\mathbb{Z}$$

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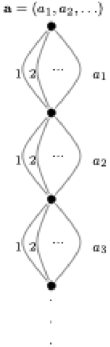
In general:

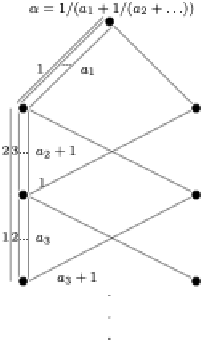
$$\text{rank}(K^0(X,T)) \leq \text{rank}_{\text{top}}(X,T).$$

Examples.

Odometer $\mathbb{Z}_{(p_n)}$, $p_n = a_1 \cdot a_2 \cdots a_{n-1}$.

Sturmain $(\hat{\mathbb{S}}^1, \hat{R}_\theta)$ with rotation number $\theta = [a_1, a_2, \dots]$.

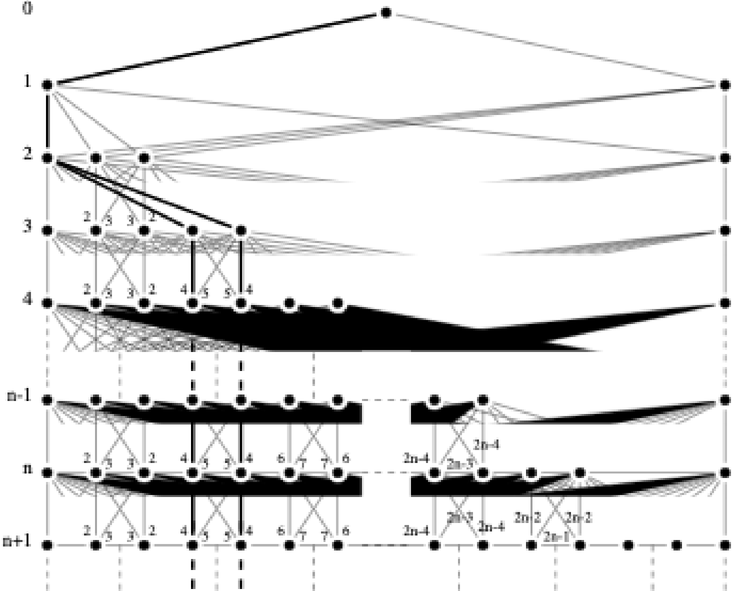




$K^0(\mathbb{Z}_{(p_n)}, +1) = \mathbb{Z}[\frac{1}{2}]$

$K^0(\hat{\mathbb{S}}^1, \hat{R}_\theta) = \mathbb{Z} + \theta\mathbb{Z}$

Non-odometer system of Algebraic rank 1.



Topological Factoring and Dimension Groups.

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$$\pi^*([f]) = [f \circ \pi].$$

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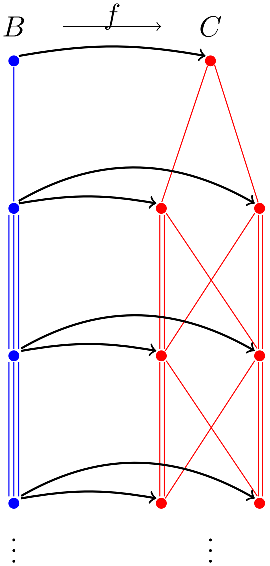
There exist morphisms between the levels of their un-ordered Bratteli diagrams: [Amini–Elliott–Golestani, 2015]

$$\begin{array}{llllllll}
B = (V,E): & V_0 & \xrightarrow{E_1} & V_1 & \xrightarrow{E_2} & V_2 & \xrightarrow{E_3} & \cdots \\
 & \downarrow F_0 & & \downarrow F_1 & & \downarrow F_2 & & \\
C = (W,E'): & W_{n_0} & \xrightarrow[E'_{n_1,n_0}]{} & W_{n_1} & \xrightarrow[E'_{n_2,n_1}]{} & W_{n_2} & \xrightarrow[E'_{n_3,n_2}]{} & \cdots \; .
\end{array}$$

Example



Factor maps and ordered premorphisms

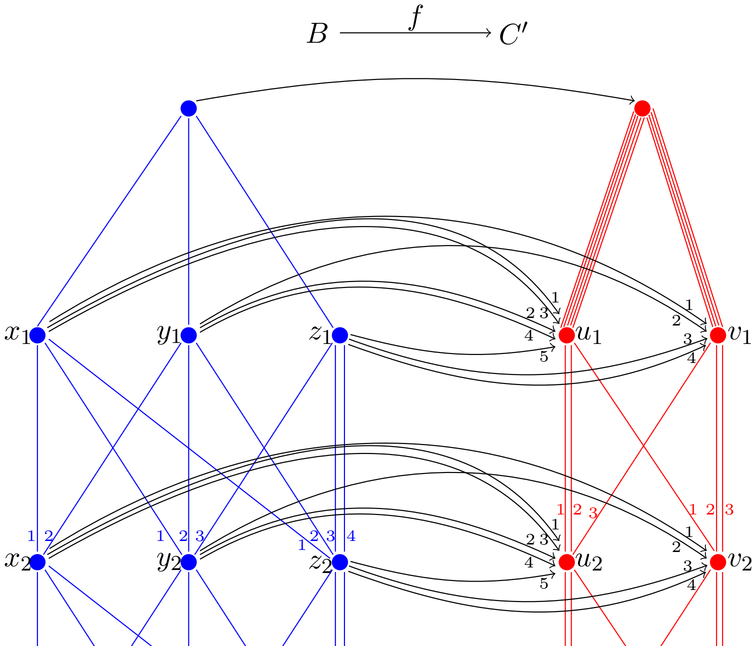
Theorem. (Amini–Elliott–Golestani, 2019)

Let (X, T) *and* (Y, S) *be Cantor minimal systems, and let* $x \in X$ *and* $y \in Y$ *. Suppose that* B_1 *and* B_2 *are Bratteli-Vershik models for* (Y, S, y) *and* (X, T, x) *respectively. The following statements are equivalent:*

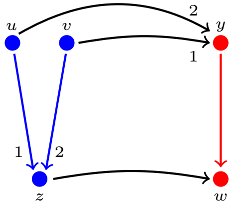
1. *there is a factor map* $\alpha : (X, T) \to (Y, S)$ *with* $\alpha(x) = y$*;*
2. *there is an ordered morphism* f *from* B_1 *to* B_2*.*

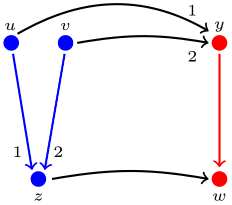
More precisely, there is a one-to-one correspondence between the set of factor maps α *as in (1) and the set of morphisms* f *from* B_1 *to* B_2*.*

Example.



Ordered commutativity vs. unordered commutativity





From Order Embedding to Topological Factoring.

Suppose G and H are non-cyclic dimension groups and

$$\iota : H \hookrightarrow G$$

is an order embedding. Does there exist Realizations (X, T) for G and (Y, S) for H with topological factoring:

$$\pi : (X, T) \longrightarrow (Y, S)?$$

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- [GPS, 2001] : If G/H is torsion free and H is order-dense in G then

$$\exists\ (X, T) \text{ and } (Y, S) \text{ with almost } 1-1\ \pi : (X, T) \longrightarrow (Y, S).$$

- [F.Sugisaki, 2011] : If G/H is torsion free and $\mathcal{S}(H)$ is affinely homomorphic to $\mathcal{S}(G)$, preserving the extreme points, then

$$\exists\ (X, T) \text{ and } (Y, S) \text{ with almost } 1-1\ \pi : (X, T) \longrightarrow (Y, S).$$

But not All realizations of G and H are intertwine with each others. For instance,

$$\mathbb{Z} + 2\theta\mathbb{Z} \quad \hookrightarrow \quad \mathbb{Z} + \theta\mathbb{Z}$$

But Sturmians are Cantor prime.

$$(\hat{\mathbb{S}}^1, \hat{R}_\theta) \quad \not\rightarrow \quad (\hat{\mathbb{S}}^1, \hat{R}_{2\theta})$$

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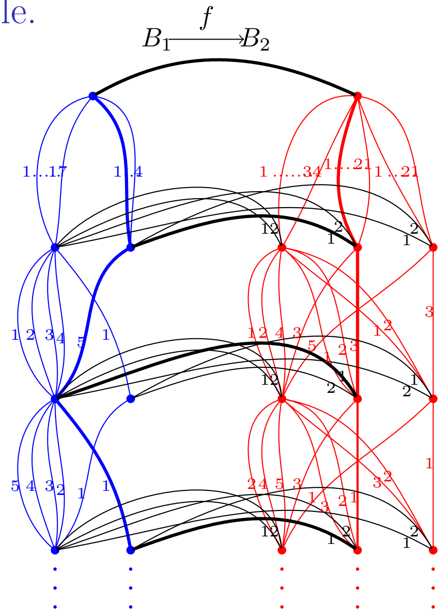
However, Using ordered morphism technique

$\exists$ an extension of $(\hat{\mathbb{S}}^1, \hat{R}_{2\theta})$ *to an orbit equivalent system to* $(\hat{\mathbb{S}}^1, \hat{R}_\theta)$.

An example.



Ordered premorphisms and the rank

Proposition. ([N. Golestani, M.H. 2021)

Let $f\colon B_1 \to B_2$ be an ordered premorphism between two properly ordered Bratteli diagrams such that B_1 is simple. Consider the the Vershik system on B_1. Then

$$\mathrm{rank}_{\mathrm{top}}(X_{B_1}, T_{B_1}) \leq 3\,\mathrm{rank}(B_2).$$

Main Ingredients of the Proof.

Lemma (Fine-Wilf theorem)

Let A be a finite alphabet and let $k \in \mathbb{N}$. If $w \in A^*$ has periods $p_1, p_2, \ldots, p_k$ such that $|w| \geq p_1 + p_2 + \cdots + p_k - \gcd(p_1, p_2, \ldots, p_k)$, then w is periodic with period $\gcd(p_1, p_2, \ldots, p_k)$.

Generalization:

Lemma (N. Golestani, M.H. 2021)

Let A be a finite alphabet and let $p \in \mathbb{N}$. Let $s_1, \ldots, s_p, t_1, \ldots, t_p$, and w be words in A^* such that

$$w = s_1 t_1 = s_2 t_2 = \cdots = s_p t_p.$$

Suppose that there are two words s and t in A^* with $|s|, |t| \geq |w|$ such that for any $1 \leq i \leq p$, s_i is a suffix of s and t_i is a prefix of t. Then there exists a set of words $B \subseteq A^*$ such that

1. $\mathrm{card}(B) \leq 3$;
2. $s_i, t_i \in B^*$ *for every* $1 \leq i \leq p$.

Theorem. (N. Golestani, M.H. 2021)

Let (X, T) be an essentially minimal Cantor system and (Y, S) be Cantor minimal system such that for some continuous map $\alpha : X \to Y$, $\alpha \circ T = S \circ \alpha$. Then

$$\mathrm{rank}_{\mathrm{top}}(Y, S) \leq 3\,\mathrm{rank}_{\mathrm{top}}(X, T).$$

In particular, if $\mathrm{rank}_{\mathrm{top}}(X, T) < \infty$ *then* $\mathrm{rank}_{\mathrm{top}}(Y, S) < \infty$.

Proof.

Choose a base point $x \in X$ for (X, T) giving a model B_2 realizing the rank of (X, T), i.e., $\mathrm{rank}_{\mathrm{top}}(X, T) = \mathrm{rank}(B_2)$. Put $y := \alpha(x) \in Y$ and let B_1 be a model for (Y, S) based on $y_{\min} := y$. We get an ordered premorphism $f : B_1 \to B_2$. Now,

$$\mathrm{rank}(Y, S) = \mathrm{rank}(X_{B_1}, T_{B_1}) \leq 3\mathrm{rank}(B_2) = 3\mathrm{rank}(X, T).$$

□

Some corollaries

Corollary

Every minimal Cantor factor of a finite topological rank subshift is an odometer or a subshift.

This dichotomy was previously proved for Vershik systems:

Theorem (T. Downarowicz, A. Maass 2000)

Every finite rank Bratteli-Vershik system is either subshift or an odometer.

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Corollary

Any primitive S-adic subshift with bounded alphabet rank is of finite topological rank.

Generalization to non-proper Brattel Diagrams.

Definition (T. Downarowicz, O. Karpel 2018)

An ordered Bratteli diagram $(B, <)$ is decisive if the Vershik map T_V prolongs in a unique way to a homeomorphism $\bar{T}_V$ of X_B. A zero-dimensional dynamical system (X, T) will be called Bratteli-Vershikizable if it is conjugate to $(X_B, \bar{T}_V)$ for a decisive ordered Bratteli diagram $(B, <)$.

Proposition. (N. Golestani, M. H.)

Let (X, T) and (Y, S) be two zero dimensional Dicisive systems. Suppose that B_1 and B_2 are Bratteli-Vershik models for (Y, S) and (X, T) respectively. The following statements are equivalent:

1. *there is a factor map $\alpha : (X, T) \to (Y, S)$;*
2. *there is an ordered morphism f from B_1 to B_2.*