G-maps over the homogeneous space G/H as equivariant fibrations

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- A continuous map $\varphi : X \longrightarrow Y$, where X and Y are G-spaces, is called a G-map or an equivariant map if $\varphi(gx) = g\varphi(x)$ for every $x \in X$ and $g \in G$.

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We work in the category G-TOP of G-spaces and G-maps.

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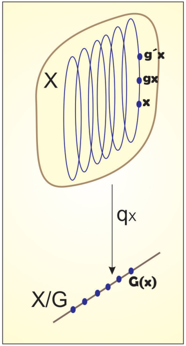
3. If H is a closed subgroup of G, G can be considered as an H-space with the conjugation action:

$$h * g = hgh^{-1}.$$

The G-subset $G(x) = \{gx \mid g \in G\}$ is called the G-orbit of x.



Given a G-space, the set of its G-orbits, endowed with the quotient topology with respect to the canonical projection $X \to X/G$, is called the G-orbit space of X and is denoted by X/G.

Let G be a topological group and H a closed subgroup of G. If X is an H-space, then we can consider $G \times X$ as an H-space with the action

$$h \cdot (g, x) = (gh^{-1}, hx).$$

The twisted product $G \times_H X$ is defined as the corresponding orbit space

$$G \times_H X = (G \times X)/H.$$

The H-orbit of the point (g, x) is denoted by $[g, x]$.

G acts on $G \times_H X$ by

$$g' \cdot [g, x] = [g'g, x],$$

hence, $G \times_H X$ is a G-space.

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Proposition (Antonyan)

Let H be a compact subgroup of an almost connected group G such that G/H is locally connected and finite-dimensional, then there exists a compact normal subgroup N of G such that $N \subset H$ and G/N is a Lie group.

A G-map $p : E \to B$ is called *G-fibration* if it has the equivariant homotopy lifting property (*EHLP*) with respect to all G-spaces X i.e. if for every commutative diagram of G-maps

$$\begin{array}{ccccc} x & \quad & X & \xrightarrow{\;h\;} & E \\ \big\downarrow & & {\scriptstyle\partial_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ (x,0) & & X \times I & \xrightarrow[\;H\;]{} & B \end{array}$$

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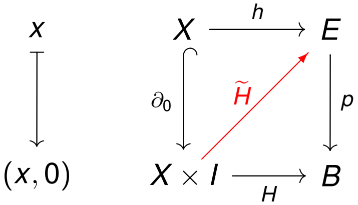

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Theorem

Let H be a closed subgroup of a compact Lie group G and E a G-space. Then any G-map $p : E \to G/H$ is a G-fibration.

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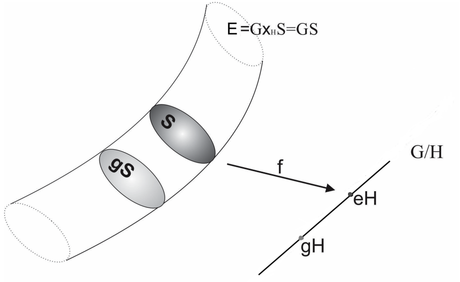
Structure of a map over G/H



Theorem (Skljarenko)

Let G be a locally compact group, H and K be closed subgroups of G such that $K \subseteq H$. The projection $q : G/K \to G/H$ has the covering homotopy property for arbitrary spaces.

Proposition

Let H be a closed subgroup of a locally compact group G. Then the projection $\pi : G \to G/H$, $g \mapsto gH$, is a G-fibration where both spaces are endowed with the actions defined by left translations.

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Corollary

Let N and H be closed subgroups of a locally compact group G such that N is a normal subgroup of G and $N \subseteq H$. Then the projection $q : G/N \to G/H$, $gN \mapsto gH$ is a G-fibration.

Lemma (Lashof)

Let H be a closed subgroup of a compact Lie group G. If G is considered as an H-space by conjugation with the action $h * g = hgh^{-1}$, and H acts on G/H by $h \cdot gH = hgH$, then the projection $q : G \to G/H$, $g \mapsto gH$, is an H-fibration.

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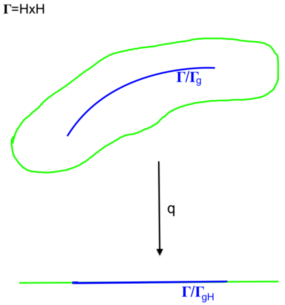
Proposition (Lashof)

Let H be a closed subgroup of a compact Lie group G. Let X be a G-space and A an H-space such that there is a G-homeomorphism $\eta : X \times I \to G \times_H A$, then A is H-homeomorphic to $A_0 \times I$, where $A_0 = X \times \{0\} \cap \eta^{-1}([e, A])$.

Lemma

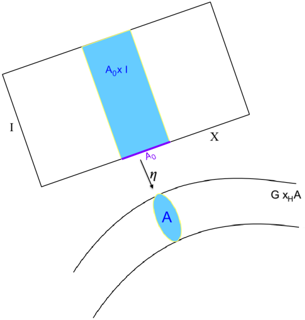
Let H be a compact subgroup of a Lie group G. If G is considered as an H-space by conjugation with the action $h * g = hgh^{-1}$, and H acts on G/H by $h \cdot gH = hgH$, then the projection $q : G \to G/H$, $g \mapsto gH$, is an H-fibration.



Proposition

Let $X \times I \to G \times_H A$ be a G-homeomorphism, then A is H-homeomorphic to $A_0 \times I$ if either of the following conditions holds:

- G is a Lie group and H is its compact subgroup.
- G is an almost connected group and H is a compact subgroup such that G/H is a smooth manifold.



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$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ {\scriptstyle \partial_0} \big\downarrow & & \big\downarrow {\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & G/H \end{array} \qquad \begin{array}{ccc} A_0 & \xrightarrow{f'} & S \\ {\scriptstyle \partial_0'} \big\downarrow & & \big\downarrow {\scriptstyle p'} \\ A_0 \times I & \xrightarrow[F']{} & \{eH\} \end{array}$$

$A_0 \times I \approx F^{-1}(eH)$ and $S = p^{-1}(eH)$.

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\begin{array}{ccc}
G \times_H A_0 & \xrightarrow{G\times_H f'} & G \times_H S \\
{\scriptstyle G\times_H \partial_0'} \big\downarrow & \overset{G\times_H \bar{F}}{\nearrow} & \big\downarrow {\scriptstyle G\times_H p'} \\
G \times_H (A_0 \times I) & \xrightarrow[G\times_H F']{} & G \times_H \{eH\}
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Theorem (Bykov)

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Let H be a closed subgroup of a compact metrizable group G. If $p : E \to B$ is an H-fibration, then the G-map $G \times_H p : G \times_H E \to G \times_H B$ is a G-fibration.

Proposition (Bykov)

Let G and K be compact Lie groups related by the homomorphism $\alpha : G \to Aut(K)$. Let E be a metrizable left G-space equipped also with a right free action of the group K such that

$$g(yk) = (gy)\alpha_g(k)$$

holds for all $g \in G$, $k \in K$, $y \in E$, and $\alpha_g = \alpha(g) : K \to K$. Then the K-orbit map $p : E \to E/K$ is a regular G-fibration, where the K-orbit space is regarded as a G-space with the action $g \cdot yK = (gy)K$.

Proposition

Let H be a compact subgroup of an almost connected metrizable group G. If G is considered as an H-space by conjugation with the action $h * g = hgh^{-1}$, and the action on G/H is given by $h \cdot gH = hgH$, then the projection $q : G \to G/H$, $g \mapsto gH$, is an H-fibration.

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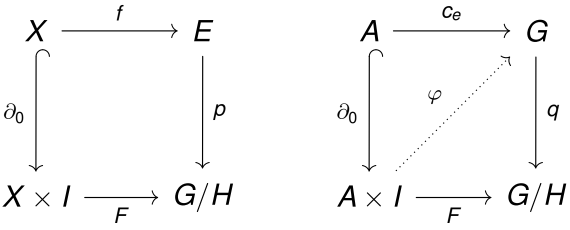

$A = f^{-1}p^{-1}(eH)$

Every homomorphism of topological groups $\alpha : G' \to G$ induces the *restriction functor*

$$res_\alpha : G\text{-}TOP \to G'\text{-}TOP.$$

If X is a G-space, it can be considered as a G'-space via α, this is, with the action $*$ of G' given by $g' * x = \alpha(g') \cdot x$, for all $g' \in G'$, $x \in X$.

And is right adjoint of the *functor of twisted product via α*:

$$G \times_\alpha - : G'\text{-}TOP \to G\text{-}TOP.$$

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In particular, if H is a closed subgroup of G, for the inclusion $i : H \hookrightarrow G$, the restriction functor $res : G\text{-}TOP \to H\text{-}TOP$, is right adjoint of the functor of twisted product $G \times_H - : H\text{-}TOP \to G\text{-}TOP$.

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Let H be a compact subgroup of a group G. If $p : E \to B$ is an H-fibration, then the G-map

$$G \times_H p : G \times_H E \to G \times_H B$$

is a G-fibration if one of the following conditions holds:

1. *G is a compact Lie group.*
2. *G is a compact metrizable group. (Bykov)*
3. *G is a Lie group.*
4. *G is an almost connected metrizable group.*

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