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## Generalized Bratteli-Vershik diagrams

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## Bratteli diagrams



A Bratteli diagram is an infinite directed graph $B=(V, E)$ :

- vertex set $V=\bigsqcup_{i \geq 0} V_{i}$,
- edge set $E=\bigsqcup_{i \geq 0} E_{i}$,
- $V_{0}=\left\{v_{0}\right\}$ is a single point,
- $V_{i}$ and $E_{i}$ are finite sets,
- edges $E_{i}$ connect $V_{i}$ to $V_{i+1}$
- every $v \in V$ has an outgoing edge and every $v \in V \backslash V_{0}$ has and incoming edge.
$V_{i}$ is called the $i$-th level of the diagram.
$X_{B}$ is the set of all infinite paths that start at $v_{0}$.


## Ordered Bratteli diagrams



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- Take a vertex $v \in V \backslash V_{0}$.


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- Take a vertex $v \in V \backslash V_{0}$.
- Consider the set of all edges that end in $v$.


## Ordered Bratteli diagrams



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- Consider the set of all edges that end in $v$.
- Enumerate edges from this set.


## Ordered Bratteli diagrams



- Take a vertex $v \in V \backslash V_{0}$.
- Consider the set of all edges that end in $v$.
- Enumerate edges from this set.
- Do the same for every vertex.


## Ordered Bratteli diagrams



## Ordered Bratteli diagrams



- An infinite path $x=\left(x_{n}\right)$ is called maximal if for every $n, x_{n}$ is maximal among all edges that end in the same vertex as $x_{n}$.


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- An infinite path $x=\left(x_{n}\right)$ is called maximal if for every $n, x_{n}$ is maximal among all edges that end in the same vertex as $x_{n}$.
- The sets $X_{\text {max }}$ and $X_{\text {min }}$ of all maximal and minimal paths are non-empty and closed.


## Vershik map



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## Vershik map

Define the Vershik map
$\varphi_{B}: X_{B} \backslash X_{\max } \rightarrow X_{B} \backslash X_{\min }:$
Fix $x=\left\{x_{k}\right\}_{k=1}^{\infty} \in X_{B} \backslash X_{\max }$.

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Fix $x=\left\{x_{k}\right\}_{k=1}^{\infty} \in X_{B} \backslash X_{\text {max }}$.
Find the first $k$ with $x_{k}$ non-maximal.

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Find the first $k$ with $x_{k}$ non-maximal.

Take the successor $\bar{x}_{k}$ of $x_{k}$.

## Vershik map

Define the Vershik map
$\varphi_{B}: X_{B} \backslash X_{\max } \rightarrow X_{B} \backslash X_{\min }:$
Fix $x=\left\{x_{k}\right\}_{k=1}^{\infty} \in X_{B} \backslash X_{\max }$.
Find the first $k$ with $x_{k}$ non-maximal.

Take the successor $\bar{x}_{k}$ of $x_{k}$. Connect $s\left(\bar{x}_{k}\right)$ to the top vertex $V_{0}$ by the minimal path.

## Vershik map

Define the Vershik map

$\varphi_{B}: X_{B} \backslash X_{\max } \rightarrow X_{B} \backslash X_{\text {min }}:$
Fix $x=\left\{x_{k}\right\}_{k=1}^{\infty} \in X_{B} \backslash X_{\max }$.
Find the first $k$ with $x_{k}$ non-maximal.

Take the successor $\bar{x}_{k}$ of $x_{k}$.
Connect $s\left(\bar{x}_{k}\right)$ to the top vertex $V_{0}$ by the minimal path.
$\varphi_{B}$ is defined everywhere on
$X_{B} \backslash X_{\text {max }}$,
$\varphi_{B}\left(X_{B} \backslash X_{\max }\right)=X_{B} \backslash X_{\text {min }}$
If the map $\varphi_{B}$ can be extended to a homeomorphism of $X_{B}$
such that $\varphi_{B}\left(X_{\max }\right)=X_{\text {min }}$, then $\left(X_{B}, \varphi_{B}\right)$ is called a
Bratteli-Vershik system and $\varphi_{B}$ is called the Vershik map.

## Motivation

- Bratteli diagrams (Bratteli, 1972): classification of $C^{*}$-algebras
- Bratteli-Vershik models
- measurable dynamics (Vershik, 1980's)
- Cantor (compact zero-dimensional) dynamics (Herman-Putnam-Skau, 1992, Medynets, 2006, Shimomura, 2018, Downarowicz-K., 2019)
- Borel dynamics (Bezuglyi-Dooley-Kwiatkowski, 2006)
- classification of Cantor dynamical systems
- Kakutani equivalence (Herman-Putnam-Skau, 1992)
- orbit equivalence (Giordano-Putnam-Skau, 1995)
- characterization of particular classes of Cantor dynamical systems (substitution dynamical systems, interval exchange transformations, Toeplitz systems, etc.)
- describing the simplex of probability invariant measures


## Invariant measures on Bratteli diagrams



Two infinite paths are called tail (cofinal) equivalent if they coincide starting from some level.

A measure $\mu$ on $X_{B}$ is called invariant if $\mu([\bar{e}])=\mu\left(\left[\bar{e}^{\prime}\right]\right)$ for any two cylinders $[\bar{e}]$ and $\left[\bar{e}^{\prime}\right]$, such that the finite paths $\bar{e}$ and $\bar{e}^{\prime}$ have the same range.

Continuous Vershik map does not always exist on a Bratteli diagram (Medynets (2006); Bezuglyi-Kwiatkowski-Yassawi (2014), Bezuglyi-Yassawi (2017))

## Incidence matrices



The $n$-th incidence matrix $F_{n}=\left(f_{v, w}^{(n)}\right), n \geq 0$, is a $\left|V_{n+1}\right| \times\left|V_{n}\right|$ matrix such that $f_{V, w}^{(n)}$ is the number of edges between $v \in V_{n+1}$ and $w \in V_{n}$.

$$
\begin{gathered}
F_{0}=\binom{1}{1}, \\
F_{1}=\left(\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right), \\
F_{2}=\left(\begin{array}{ll}
1 & 1 \\
1 & 3 \\
1 & 1
\end{array}\right) .
\end{gathered}
$$

## Stationary Bratteli diagrams

A Bratteli diagram is called stationary if $F_{n}=F$ for all $n \geq 1$.
There is a one-to-one correspondence between non-negative (right) eigenvectors of $A=F^{T}$ and finite ergodic invariant measures on $X_{B}$ (Bezuglyi-Kwiatkowski-Medynets-Solomyak, 2010).

Let $A x=\lambda x$, where $x$ is a non-negative probability vector. Then the corresponding measure $\mu$ satisfies the relation:

$$
p_{w}^{(n)}=\frac{x_{w}}{\lambda^{n-1}}
$$

where $p_{w}^{(n)}$ is a measure of a cylinder set corresponding to a finite path between $v_{0}$ and $w \in V_{n}$.

## Example



$$
\begin{gathered}
A=F^{T}=\left(\begin{array}{lll}
1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 0 & 3
\end{array}\right) \\
\lambda^{(1)}=\frac{3+\sqrt{5}}{2} \\
x^{(1)}=\left(\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}, 0\right)^{T}, \\
\lambda^{(2)}=3, x^{(2)}=\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^{T} .
\end{gathered}
$$

## Example



$$
\begin{gathered}
A=F^{T}=\left(\begin{array}{lll}
1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 0 & 3
\end{array}\right) \\
\lambda=3 \\
x=\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^{T}
\end{gathered}
$$

## Example



$$
\begin{gathered}
A=\left(\begin{array}{lll}
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\end{gathered}
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## Example



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\begin{gathered}
A=\left(\begin{array}{lll}
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\end{array}\right) \\
\lambda=3 \\
x=\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^{T}
\end{gathered}
$$

A Borel dynamical system is a pair $(X, T)$, where $X=(X, \mathcal{B})$ is a (uncountable) standard Borel space with the $\sigma$-algebra $\mathcal{B}$ of Borel sets, and $T$ is a Borel automorphism of $X$.

A generalized Bratteli diagram is an infinite directed graph
$B=(V, E)$ such that:

- vertex set $V=\bigsqcup_{i \geq 0} V_{i}$, edge set $E=\bigsqcup_{i \geq 0} E_{i}$,
- $V_{0}=V_{i}$ is an infinite countable set for all $i \geq 1$,
- edges $E_{i}$ connect $V_{i}$ to $V_{i+1}$,
- every $w \in V$ has an outgoing edge $e(s(e)=w)$ and every $v \in V \backslash V_{0}$ has and incoming edge $e^{\prime}\left(r\left(e^{\prime}\right)=v\right)$,
- for every $w \in V_{i}$ and $v \in V_{i+1}$, the set of edges $E(w, v)$ between $w$ and $v$ is finite, $|E(w, v)|=f_{v w}^{(i)}$,
- matrices $F_{i}$ have only finitely many non-zero entries in each row (for any $v \in V \backslash V_{0}, r^{-1}(v)$ is finite).


Figure: Example of a generalized Bratteli diagram

Theorem (Bezuglyi-Dooley-Kwiatkowski, 2006)
Every aperiodic Borel automorphism of a standard Borel space is isomorphic to the Vershik map acting on the path space of an ordered generalized Bratteli diagram.

## Perron-Frobenius Theory for infinite matrices

A non-negative infinite matrix $A=\left(a_{i j}\right)$ is called irreducible if for any $i, j$ there exists $n>0$ such that $\left(A^{n}\right)_{i j}>0$.
Fix $i \in \mathbb{Z}$, let $p(i)=\operatorname{gcd}\left\{n:\left(A^{n}\right)_{i i}>0\right\}$. If $A$ is irreducible then $p(i)=p$ for all $i$. If $p=1$ then $A$ is aperiodic.
Lemma
Let $A$ be a real, non-negative, irreducible and aperiodic infinite matrix. Then for all $i, \lambda=\lim _{n \rightarrow \infty} \sqrt[n]{\left(A^{n}\right)_{i i}}=\sup _{n \in \mathbb{N}} \sqrt[n]{\left(A^{n}\right)_{i i}} \leq \infty$.
$\lambda$ is called the Perron eigenvalue of $A$. Assume $\lambda<\infty$.

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \frac{\left(A^{n}\right)_{i i}}{\lambda^{n}}=\infty \Longleftrightarrow A \text { is recurrent } \\
& \sum_{n=0}^{\infty} \frac{\left(A^{n}\right)_{i i}}{\lambda^{n}}<\infty \Longleftrightarrow A \text { is transient }
\end{aligned}
$$

## Perron-Frobenius Theory for infinite matrices

## Theorem (Generalized Perron-Frobenius theorem)

Let A be a real, non-negative, irreducible, aperiodic and recurrent infinite matrix with Perron eigenvalue $\lambda$. Then

- there exist strictly positive eigenvectors $\eta, \xi$ such that

$$
\eta \boldsymbol{A}=\lambda \eta, \boldsymbol{A} \xi=\lambda \xi
$$

- $\eta$ and $\xi$ are unique up to constant multiples;
- $\eta \cdot \xi<\infty \Longleftrightarrow A$ is positive recurrent
- if $A$ is positive recurrent and $\eta \xi=1$ then

$$
\lim _{n \rightarrow \infty} \frac{A^{n}}{\lambda^{n}}=\xi \eta
$$

## Theorem (Bezuglyi-Jorgensen, 2021)

Let $B$ be a stationary generalized Bratteli diagram with irreducible, aperiodic and recurrent matrix A transpose to the incidence matrix. Then

1. there exists a tail invariant measure $\mu$ on the path space $X_{B}$, defined as follows: let $\bar{e}(w, v)$ denote a finite path that begins at $w \in V_{0}$ and ends at $v \in V_{n}, n \in \mathbb{N}$. For the corresponding cylinder set $[\bar{e}(w, v)]$, we set

$$
\mu([\bar{e}(w, v)])=\frac{\xi_{v}}{\lambda^{n}} .
$$

Here $\xi=\left(\xi_{v}\right)$ is the right eigenvector corresponding to the Perron eigenvalue $\lambda$ for $A$;
2. measure $\mu$ is finite if and only if

$$
\sum_{v} \xi_{v}<\infty
$$

## Invariant measures for generalized Bratteli diagrams

## Theorem (Bezuglyi-Jorgensen-K.-Sanadhya)

Let $B$ be an ordered stationary generalized Bratteli diagram with the matrix $A=F^{\top}$ transpose to the incidence matrix and a Vershik map $\varphi_{B}$. Let $A$ be irreducible, aperiodic and positive recurrent. Let $\lambda$ be the Perron eigenvalue for $A$ and $\xi=\left(\xi_{i}\right)$ be the corresponding right eigenvector with $\sum \xi_{i}=1$.
Then the measure $\mu$ defined as follows: for a cylinder set $[\bar{e}]=\left(e_{0}, \ldots, e_{n-1}\right)$ with $r\left(e_{n-1}\right)=w$ :

$$
\mu([\bar{e}])=\frac{\xi_{w}}{\lambda^{n}}
$$

is a unique probability invariant measure for $\varphi_{B}$ which is positive on cylinder sets.

## Sketch of proof

Suppose $\nu$ is a probability ergodic $\varphi_{B}$-invariant measure which is positive on cylinder sets. Denote by $p_{w}^{(n)}$ the measure $\nu$ of a cylinder set $[\bar{e}]=\left(e_{0}, \ldots, e_{n-1}\right)$ with $r\left(e_{n-1}\right)=w$. Then by Birkhoff ergodic theorem,

$$
p_{w}^{(n)}=\lim _{N \rightarrow \infty} \frac{|E(w, v)|}{h_{v}^{(N)}}
$$

Then using Perron-Frobenius theorem we obtain

$$
\nu([\bar{e}])=p_{w}^{(n)}=\lim _{N \rightarrow \infty} \frac{\left(A^{N-n}\right)_{w v}}{\sum_{u \in V_{0}}\left(A^{N}\right)_{u v}}=\frac{\xi_{w} \cdot \eta_{v}}{\sum_{u \in V_{0}} \xi_{u} \cdot \eta_{v} \cdot \lambda^{n}}=\frac{\xi_{w}}{\lambda^{n}}=\mu([\bar{e}])
$$



## Invariant measures for generalized Bratteli diagrams

## Theorem (Bezuglyi-Jorgensen-K.-Sanadhya)

Let $B$ be an ordered stationary generalized Bratteli diagram with matrix $A=F^{\top}$ transpose to the incidence matrix and Vershik map $\varphi_{B}$. Let $A$ be irreducible, aperiodic and positive recurrent. Let $\lambda$ be a Perron eigenvalue of $A$ and $\xi=\left(\xi_{i}\right)$ be the corresponding right eigenvector with $\sum \xi_{i}=\infty$.
Let $\mu$ be an ergodic $\varphi_{B}$-invariant infinite $\sigma$-finite measure such that $\left(X, \mathcal{B}, \varphi_{B}, \mu\right)$ is conservative and $\mu$ takes finite positive values on cylinder sets. Then $\mu$ is unique up to a constant multiple and can be defined as follows: for a cylinder set $[\bar{e}]=\left(e_{0}, \ldots, e_{n-1}\right)$ with $r\left(e_{n-1}\right)=w$ :

$$
\mu([\bar{e}])=\frac{\xi_{w}}{\lambda^{n}} .
$$

## Sketch of proof

Consider two cylinder sets $\left[\overline{e_{1}}\right],\left[\overline{e_{2}}\right] \subset X_{B}$ such that $r\left(\overline{e_{1}}\right)=w_{1} \in V_{n_{1}}$ and $r\left(\overline{e_{2}}\right)=w_{2} \in V_{n_{2}}$.
For some $N>\max \left\{n_{1}, n_{2}\right\}$, let $v \in V_{N}$ such that the sets $E\left(w_{1}, v\right)$ and $E\left(w_{2}, v\right)$ are non-empty. By Hopf's ratio ergodic theorem,

$$
\frac{m\left(\left[\overline{e_{1}}\right]\right)}{m\left(\left[\overline{e_{2}}\right]\right)}=\lim _{N \rightarrow \infty} \frac{\left|E\left(w_{1}, v\right)\right|}{\left|E\left(w_{2}, v\right)\right|}=\lim _{N \rightarrow \infty} \frac{A_{w_{1} v}^{\left(N-n_{1}\right)}}{A_{w_{2} v}^{\left(N-n_{2}\right)}}=\frac{\xi_{w_{1}}}{\xi_{w_{2}}} \lambda^{\left(n_{2}-n_{1}\right)} .
$$

$$
A=F^{T}=\left(\begin{array}{cccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & 2 b & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 2 b & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 2 b & 0 & b & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 2 b & a & b & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & b & a & 2 b & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & b & 0 & 2 b & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 2 b & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 2 b & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
$$

$$
A \xi=\lambda \xi, \quad \lambda=a+2 b
$$

$$
\xi=\left(\ldots, \frac{1}{2^{3}}\left(\frac{a}{b}\right)^{2}, \frac{1}{2^{2}}\left(\frac{a}{b}\right), \frac{1}{2}, 1,1, \frac{1}{2}, \frac{1}{2^{2}}\left(\frac{a}{b}\right), \frac{1}{2^{3}}\left(\frac{a}{b}\right)^{2}, \ldots\right)^{T}
$$

$$
\sum_{i \in \mathbb{Z}} \xi_{i}<\infty \Longleftrightarrow a<2 b
$$

## Example (Bobok-Bruin, 2016)

$$
\begin{gathered}
A=F^{T}=\left(\begin{array}{ccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdot \\
\cdots & 0 & b & 0 & 0 & 0 & \cdots \\
\cdots & a & 0 & b & 0 & 0 & \cdots \\
\cdots & 0 & a & 0 & b & 0 & \cdots \\
\cdots & 0 & 0 & a & 0 & b & \cdots \\
\cdots & 0 & 0 & 0 & a & 0 & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right) \\
\boldsymbol{A} \xi=\lambda \xi, \quad \lambda=a+b, \quad \xi=\left(\xi_{i}\right)=(\ldots, 1,1,1 \ldots)^{T}, \quad \sum_{i \in \mathbb{Z}} \xi_{i}=\infty ; \\
A x=\lambda_{A} x, \quad \lambda_{A}=2 \sqrt{a b}, \quad x=\left(x_{i}\right), \quad x_{i}=\left(\frac{a}{b}\right)^{\frac{i}{2}}, \quad \sum_{i \in \mathbb{Z}} x_{i}=\infty .
\end{gathered}
$$

## Stochastic matrices and random walks

Assume that the matrix $A$ is a real non-negative irreducible and aperiodic infinite matrix and there exists a positive right eigenvector $\xi$ for some $\lambda<\infty$ :

$$
\boldsymbol{A} \xi=\lambda \xi
$$

Define the matrix $P=\left(p_{w, v}\right)_{w, v \in V}$ as follows:

$$
p_{w, v}=\frac{a_{w, v} \xi_{v}}{\lambda \xi_{w}}
$$

It is easy to see that matrix $P$ is row stochastic.
Theorem (BJKS, Thiago Costa Raszeja)
$A$ is positive recurrent (null recurrent, transient) if and only if $P$ is positive recurrent (null recurrent, transient).

## Random walks

For the example by Bobok-Bruin: for all $k \in \mathbb{Z}$ we have

$$
p_{k, k-1}=\frac{a}{a+b} ; \quad p_{k, k+1}=\frac{b}{a+b}
$$

All other entries of $P$ are zero.
The matrices $P$ and $A$ are transient for $a \neq b$ and null recurrent for $a=b$.

## Random walks

For the example with $\lambda=a+2 b$ and

$$
\begin{gathered}
\xi=\left(\ldots, \frac{1}{2^{3}}\left(\frac{a}{b}\right)^{2}, \frac{1}{2^{2}}\left(\frac{a}{b}\right), \frac{1}{2}, 1,1, \frac{1}{2}, \frac{1}{2^{2}}\left(\frac{a}{b}\right), \frac{1}{2^{3}}\left(\frac{a}{b}\right)^{2}, \ldots\right)^{T}: \\
\quad p_{0,0}=p_{-1,-1}=\frac{a}{a+2 b} ; \quad p_{0,-1}=p_{0,1}=p_{-1,-2}=p_{-1,0}=\frac{b}{a+2 b}
\end{gathered}
$$

for $k \geq 1$ we have

$$
p_{k, k-1}=\frac{2 b}{a+2 b} ; \quad p_{k, k+1}=\frac{a}{a+2 b}
$$

and for $k \leq-2$ :

$$
p_{k, k-1}=\frac{a}{a+2 b} ; \quad p_{k, k+1}=\frac{2 b}{a+2 b} .
$$

All other entries of $P$ are zero.


Figure: A generalized Bratteli diagram with no finite ergodic invariant measure.


Let $h_{w}^{(n)}$ be the number of all finite paths between $v_{0}$ and $w \in V_{n}$ (the "height").
Then

$$
h_{v}^{(n+1)}=\sum_{w \in V_{n}} f_{v, w}^{(n)} h_{w}^{(n)} .
$$

$X_{w}^{(n)}$ is the set of all infinite paths which pass through $w \in V_{n}$ (the "tower").

## Theorem (Bezuglyi-Jorgensen-K.-Sanadhya)

Let $B$ be a stationary generalized Bratteli diagram with the matrix $A=F^{T}$ transpose to the incidence matrix. Let $A$ be irreducible, aperiodic and positive recurrent. Let $\lambda$ be the Perron eigenvalue of $A$ and $\xi=\left(\xi_{i}\right), \eta=\left(\eta_{i}\right)$ be the corresponding right and left eigenvectors normalized such that $\sum \xi_{i}=1$ and $\eta \xi=1$.

Then for every $n \in \mathbb{N}$, every $w \in V_{n}$ and every $v \in V_{n+1}$

$$
\mu\left(X_{w}^{(n)}\right) \rightarrow \eta_{w} \xi_{w} \text { as } n \rightarrow \infty
$$

and

$$
\frac{h_{w}^{(n)}}{h_{v}^{(n+1)}} \rightarrow \frac{\eta_{w}}{\lambda \eta_{v}} \text { as } n \rightarrow \infty
$$

## Proof.

$$
\begin{gathered}
\mu\left(X_{w}^{(n)}\right)=\frac{\xi_{w}}{\lambda^{n}} \sum_{t} a_{t, w}^{(n)}=\xi_{w} \sum_{t} \frac{a_{t, w}^{(n)}}{\lambda^{n}} \xrightarrow[n \rightarrow \infty]{\longrightarrow} \xi_{w} \sum_{t} \xi_{t} \eta_{w}=\xi_{w} \eta_{w} . \\
\frac{h_{w}^{(n)}}{h_{v}^{(n+1)}}=\frac{\sum_{t} a_{t w}^{(n)}}{\sum_{t} a_{t v}^{(n+1)}}=\frac{\sum_{t} a_{t v}^{(n)} \lambda^{n+1}}{\sum_{t} a_{t v}^{(n+1)} \lambda^{n} \lambda} \xrightarrow[n \rightarrow \infty]{\longrightarrow} \frac{\eta_{w}}{\lambda \eta_{v}} .
\end{gathered}
$$

## Bratteli diagrams of bounded size

## Definition

A generalized Bratteli diagram $B$ with incidence matrices
$F_{n}=\left(f_{v w}^{(n)}\right)$ is called of bounded size if there exist a sequence of pairs of natural numbers $\left(t_{n}, L_{n}\right)$ such that for all $n \in \mathbb{N}_{0}$ and all $v \in V_{n+1}$

$$
s\left(r^{-1}(v)\right) \in\left\{v-t_{n}, \ldots, v+t_{n}\right\} \quad \text { and } \quad \sum_{w \in V_{n}} f_{v w}^{(n)} \leq L_{n}
$$



We will assume that $E\left(v-t_{n}, v\right)$ and $E\left(v+t_{n}, v\right)$ are non-empty.

## Bratteli diagrams of bounded size

Let $B$ be a generalized Bratteli diagram of bounded size and $w \in V_{n}$. Then all paths passing through $w$ lie inside a subdiagram of the form of a "cone" with $v$ being the vertex of the cone.


## Bratteli diagrams of bounded size

Fix $w \in V_{0}$. Define a slanting set $Z_{w}^{+}$:

$$
Z_{w}^{+}=\left\{x=\left(x_{n}\right) \in X_{B}: s\left(x_{0}\right) \geq w \text { and } r\left(x_{n}\right) \geq w+\sum_{i=0}^{n} t_{i} \text { for } n \in \mathbb{N}_{0}\right\}
$$



Theorem (Bezuglyi-Jorgensen-K.-Sanadhya)
The slanting sets $Z_{w}^{+}, Z_{w}^{-}$are $\mathcal{R}$-invariant closed nowhere dense sets with respect to the topology generated by cylinder sets.

## Topological properties of tail equivalence relation

Theorem (Bezuglyi-Jorgensen-K.-Sanadhya)
Let $B$ be a generalized stationary Bratteli diagram with an irreducible aperiodic incidence matrix $F=\left(f_{i j}\right)_{i, j \in \mathbb{Z}}$. Then the tail equivalence relation $\mathcal{R}$ is topologically transitive.
Moreover, if $B$ is of bounded size then $\mathcal{R}$ is not minimal.
Idea of the proof.
A generalized stationary Bratteli diagram with an irreducible aperiodic incidence matrix always has "vertical" paths, which are topologically transitive points.
For generalized Bratteli diagrams of bounded size, the "slanting" paths do not have dense orbits.

For all $i, j$ there exists I such that $\left(F^{\prime}\right)_{i j}>0$.
For all $j$ there exists $s=s(j)$ such that $\left(F^{m}\right)_{j j}>0$ for all $m \geq s$.


## Thank you for

your attention!

## Frobenius normal form

Let $B$ be a stationary Bratteli diagram and $A$ the matrix transpose to the incidence matrix of $B$. Then $A$ can be transformed to the Frobenius normal form:

$$
A=\left(\begin{array}{ccccccc}
A_{1} & 0 & \cdots & 0 & Y_{1, s+1} & \cdots & Y_{1, m} \\
0 & A_{2} & \cdots & 0 & Y_{2, s+1} & \cdots & Y_{2, m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & A_{s} & Y_{s, s+1} & \cdots & Y_{s, m} \\
0 & 0 & \cdots & 0 & A_{s+1} & \cdots & Y_{s+1, m} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & A_{m}
\end{array}\right)
$$

where all $A_{i}$ are primitive matrices, $A_{1}, \ldots, A_{s}$ determine minimal components of $\mathcal{R}$, non-zero matrices $Y_{i, j}$ show how non-minimal components "interact" with minimal ones.

## Stochastic incidence matrices



Let $h_{w}^{(n)}$ be the number of all finite paths between $v_{0}$ and $w \in V_{n}$ ("height").

## Then

$$
h_{v}^{(n+1)}=\sum_{w \in V_{n}} f_{v, w}^{(n)} h_{w}^{(n)}
$$

A stochastic incidence matrix $\widetilde{F}_{n}$ :

$$
\widetilde{f}_{v w}^{(n)}=\frac{f_{v w}^{(n)} h_{w}^{(n)}}{h_{v}^{(n+1)}},
$$

## Invariant measures on Bratteli diagrams

## Theorem (Bezuglyi-Karpel-Kwiatkowski, 2019)

A Bratteli diagram $B=(V, E)$ is uniquely ergodic if and only if there exists a telescoping $B^{\prime}$ of $B$ such that

$$
\lim _{n \rightarrow \infty} \max _{V, v^{\prime} \in V_{n+1}}\left(\sum_{w \in V_{n}}\left|\widetilde{f}_{v w}^{(n)}-\widetilde{f}_{v^{\prime} w}^{(n)}\right|\right)=0
$$

where $f_{v w}^{(n)}$ are the entries of the stochastic matrix $\widetilde{F}_{n}$ defined by the diagram $B^{\prime}$.
(recall the formula $p^{(n)}=F_{n}^{T} p^{(n+1)}$, where $p^{(n)}=\left(p_{w}^{(n)}: w \in V_{n}\right)$ and $p_{w}^{(n)}$ is a measure of a cylinder set corresponding to a finite path between $v_{0}$ and $w \in V_{n}$.)

## Invariant measures on Bratteli diagrams

Theorem (Adamska-Bezuglyi-K.-Kwiatkowski, 2017) Let $B=(V, E)$ be a Bratteli diagram of rank $k$ with incidence matrices $F_{n}=\left(f_{v w}^{(n)}\right)_{v \in V_{n+1}, w \in V_{n}}$ such that for every $n$ and every $v \in V_{n+1}$ we have $\sum_{w \in V_{n}} f_{v w}^{(n)}=r_{n}$ with $r_{n} \geq 2$. Let $\operatorname{det} F_{n} \neq 0$ for every $n$ and denote

$$
z^{(n)}=\operatorname{det}\left(\begin{array}{cccc}
\frac{f_{1,1}^{(n)}}{r_{n}} & \ldots & \frac{f_{1, k-1}^{(n)}}{r_{n}} & 1 \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & f_{k, 1}^{(n)} \\
\frac{f_{n}}{r_{n}} & \cdots & \frac{f_{, k-k-1}^{(n)}}{r_{n}} & 1
\end{array}\right) .
$$

Then there exist exactly $k$ ergodic invariant measures on $B$ if and only if

$$
\prod_{n=1}^{\infty}\left|z^{(n)}\right|>0 .
$$

## Bratteli-Vershik models for homeomorphisms of a Cantor set

A Bratteli diagram is called simple if for any level $n$ there exists $m>n$ such that each pair of vertices $(v, w) \in\left(V_{n}, V_{m}\right)$ is connected by a finite path.

Theorem (Herman-Putnam-Skau, 1992)
Every minimal homeomorphism of a Cantor space can be represented as a Vershik map acting on the path space of an ordered simple Bratteli diagram, which has a unique minimal and a unique maximal paths.

## Bratteli-Vershik models for homeomorphisms of a Cantor set

- K. Medynets (2006): Bratteli-Vershik models for aperiodic homeomorphisms of a Cantor space;
- T. Shimomura (2020): Bratteli-Vershik models for arbitrary zero-dimensional dynamical systems.

Theorem (Downarowicz-K, 2019)
A (compact, invertible) zero-dimensional system $(X, T)$ is
"Bratteli-Vershikizable" (i.e. $\varphi_{B}$ can be prolonged uniquely to
$X_{\max }$ ) if and only if the set of aperiodic points is dense, or its
closure misses one periodic orbit.

## Theorem (Adamska-Bezuglyi-K.-Kwiatkowski, 2017)

Let $B$ be a Bratteli diagram with $2 \times 2$ incidence matrices $F_{n}$ such that

$$
F_{n}=\left(\begin{array}{ll}
a_{n} & c_{n} \\
d_{n} & b_{n}
\end{array}\right),
$$

where $a_{n}+c_{n}=d_{n}+b_{n}=r_{n}$ for every $n$. Then
(1) There are two finite ergodic invariant measures if and only if

$$
\sum_{n=1}^{\infty}\left(1-\frac{\left|a_{n}-d_{n}\right|}{r_{n}}\right)<\infty,
$$

In this case, one can point out explicitly the subdiagrams that support these measures.
(2) There is a unique invariant measure $\mu$ on B if and only if

$$
\sum_{n=1}^{\infty}\left(1-\frac{\left|a_{n}-d_{n}\right|}{r_{n}}\right)=\infty .
$$

