Commuting symplectomorphisms and the flux homomorphism

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Basic setting

(M, ω) : sympl. mfd

$\mathrm{Symp}(M,\omega) :=$
$\{\phi \in \mathrm{Diff}^c(M) \,;\, \phi^*\omega = \omega\}$
(the grp of symplectomorphisms)

$\mathrm{Symp}_0(M, \omega)$
:= (the identity component
of $\mathrm{Symp}(M,\omega)$)

Easy Observation

The following pairs (f, g) of symplectomorphisms satisfy $fg = gf$. Symp$_0(M, \omega)$

① $f = \varphi_X^{t_1}$, $g = \varphi_X^{t_2}$

② $\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \emptyset$

→ We will generalize

Thm ✡ (K–Kimura–Matsushita–Mimura)

(S,ω) : closed ori. surf with ω

& $\ell :=$ genus ≥ 2.

Then

$\forall f, \forall g \in \mathrm{Symp}_0(S,\omega)$ s.t. $fg = gf$,

$$\underset{\substack{\cap \\ H^1(S;\mathbb{R})}}{\mathrm{Flux}(f)} \cup \underset{\substack{\cap \\ H^1(S;\mathbb{R})}}{\mathrm{Flux}(g)} = \underset{\substack{\in \\ H^2(S;\mathbb{R}) \\ \cong \\ \mathbb{R}}}{0}.$$

Rem $\exists$ C^0-version of Thm ✡

(Ask me later)

What is Flux?

(S, ω) : as above

We define **the flux homomorphism**

$$\mathrm{Flux} : \mathrm{Symp}_0(S, \omega) \to H^1(S; \mathbb{R})$$

by

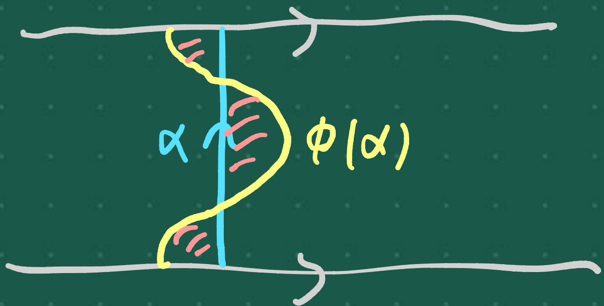
$$\varphi_X^1 \longmapsto \int_0^1 [\iota_{X_t} \omega]\, dt$$

Geometric interpretation

$\alpha : S^1 \to S$, $[\alpha] \in H_1(S;\mathbb{R})$,

$\phi \in \mathrm{Symp}_0(S)$,

$\rightsquigarrow$ $\mathrm{Flux}(\phi) \in H^1(S;\mathbb{R})$



$\subset S$

$\mathrm{Flux}(\phi)([\alpha])$

$=$ Area of 

Rem The conditions

① $f = \varphi_X^{t_1}$, $g = \varphi_X^{t_2}$

② $\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \emptyset$

implies

$$\mathrm{Flux}(f) \vee \mathrm{Flux}(g) = 0.$$

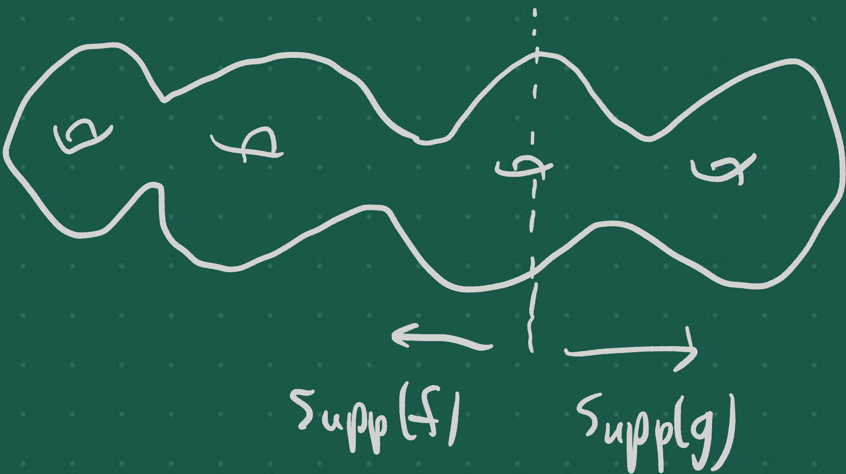
Thus, Thm ☆ is a generalization of the upper observation

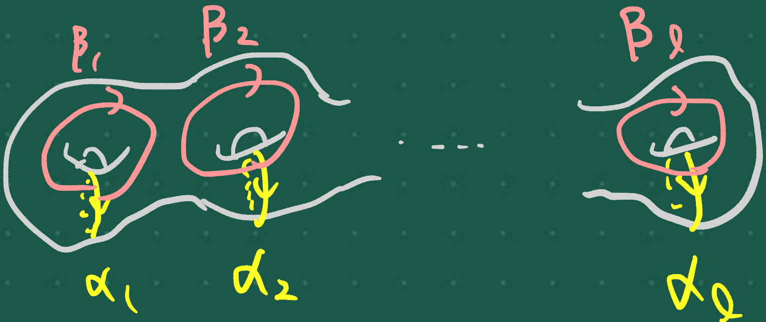
(1) $\Rightarrow \mathrm{Flux}(f) \vee \mathrm{Flux}(g) = 0.$

$\because$ (1) implies

$$\mathrm{Flux}(g) = \frac{t_2}{t_1}\,\mathrm{Flux}(f)$$

(2) $\Rightarrow \mathrm{Flux}(f) \vee \mathrm{Flux}(g) = 0$





$\alpha_1^*, \dots, \alpha_\ell^*, \beta_1^*, \dots, \beta_\ell^* \in H^1(S;\mathbb{R})$:

The dual basis of the symplectic basis $\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_\ell \in H_1(S;\mathbb{R})$.

Let

$$\mathrm{Flux}(f) = x_1 \alpha_1^* + \cdots + x_\ell \alpha_\ell^* + y_1 \beta_1^* + \cdots + y_\ell \beta_\ell^*,$$

$$\mathrm{Flux}(g) = x_1' \alpha_1^* + \cdots + x_\ell' \alpha_\ell^* + y_1' \beta_1^* + \cdots + y_\ell' \beta_\ell.$$

Note that

$$\mathrm{Flux}(f) \vee \mathrm{Flux}(g)$$

$$= \sum_i (x_i y_i' - x_i' y_i).$$

If $x_i \neq 0$, then, since

$\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \emptyset$, $y_i' \neq 0$.

How to prove Thm ✡?

Thm ✡ is an unexpected application of the following result on the extension problem of quasi-morphism on $\mathrm{Ham}(M, \omega)$

$:= \ker(\mathrm{Flux})$.

called the group of Hamiltonian diffeomorphism.

· On quasi-morphism

<u>Def</u> G: group, $\mu: G \to \mathbb{R}$

μ is called a quasi-morphism (q-m)

if $\exists C \geq 0$ s.t. $\forall x, \forall y \in G$,

$$|\mu(xy) - \mu(x) - \mu(y)| \leq C.$$

μ: q-m is called homogeneous

if $\forall x \in G, \forall n \in \mathbb{Z}$

$$\mu(x^n) = n\mu(x).$$

$Q(G) := \{\mu: G \to \mathbb{R},\ \text{homog. q-m}\}$

In 2006, Py constructed a homog. quasi-morphism

$$\mu_P : \mathrm{Ham}(S, \omega) \to \mathbb{R}$$

called Py's Calabi q-m.

Very rough idea of construction

$\partial \tilde{S} \approx S^1$

$\to$ consider some analogue of rotation q-m :

$$\widetilde{\mathrm{Homeo}}^+(S^1) \to \mathbb{R}$$

Key Thm (K^{2}M^2) $\quad (S,\omega)$: as above

Let

$\bar{v}, \bar{w} \in H^1(S;\mathbb{R})$ s.t. $\bar{v} \cup \bar{w} \neq 0$.

For $k \in \mathbb{Z}_{>0}$,

$$\Lambda_k := \left\langle \bar{v}, \frac{\bar{w}}{k} \right\rangle \subset H^1(S;\mathbb{R})$$

$$G_k := \mathrm{Flux}^{-1}(\Lambda_k).$$

Then, $\exists\, k_0 \in \mathbb{Z}_{>0}$ s.t.

$\forall k > k_0$,

$\mu_P : \mathrm{Ham}(M,\omega) \to \mathbb{R}$ is

non-extendable to G_k

(i.e. $\nexists\, \hat{\mu} \in Q(G_k)$ s.t.

$\hat{\mu}|_{\mathrm{Ham}(S,\omega)} = \mu_P$.)

Rem   K-Kimura proved that μ_P is non-ext to $\mathrm{Symp}_0(S)$.

(2022, Israel ✡ J. of Math)

· Outline of Thm 1 $\Rightarrow$ Main Thm

Take $f, g \in G_1$ s.t. $fg = gf$.

Then, we can construct a **virtual section** of

$$\mathrm{Flux} \,;\, G_K \to \Lambda_K$$

$\Longrightarrow$ μ_p is extendable to G_K

(other result by $K^2 M^2$)

$$\Longrightarrow \quad \mathrm{Flux}(f) \vee \mathrm{Flux}(g) = 0$$

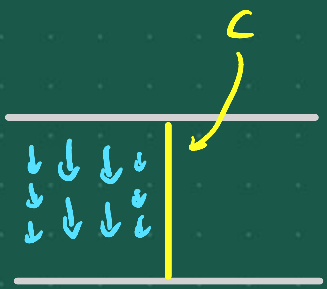
contraposition of **Thm 1**

• On the prf of Thm 1

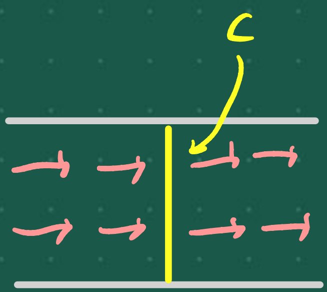
Use the following

$f, g \in \mathrm{Symp}_0(\Sigma, \omega)$

C: non-separating non-contractible curve



f: generated by the blue vector field



g: near C, parallel transf.

Then, since $f(gf^{-1}g^{-1}) = (gf^{-1}g^{-1})f$,

$[f,g]^n = [f^n,g]$.

On the other hand,

since $\mu_p([f,g]) \neq 0$,

$\mu_p([f,g]^n)$

$= n\,\mu_p([f,g]) \to +\infty$

as $n \to +\infty$

very important property of μ_p!!

Thank you for
your attension!

Danke schön!