Ellis semigroup for constant length substitutions

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Context

A tiling $\mathcal{L} \subset \mathbb{R}^d$ defines a topological dynamical system $(X_\mathcal{L}, \mathbb{R}^d)$:
the translation action of $\mathbb{R}^d$ on the hull of $\mathcal{L}$.

A symbolic sequence $\omega \in \mathcal{A}^\mathbb{Z}$ defines a topological dynamical system $(X_\omega, \mathbb{Z})$:
the shift action of $\mathbb{Z}$ on the hull of ω.

A topological dynamical system (X, T) defines a semigroup $E(X, T)$:
the completion of the group action in $\mathcal{F}(X) = \{f : X \to X\}$

The algebraic and topological properties of $E(X, T)$ caracterise (X, T) and $\mathcal{L}$ (or ω).
$E(X, T)$ is a conjugacy invariant (but not of strong orbit equivalence!) it provides information complementary to dimension groups (cohomology, or K-theory).

That is why I want to understand better $E(X, T)$.
Here I focus on constant length substitution sequences (so $T = \mathbb{Z}$).

Three compactifications of T from an aperiodic tiling/sequence

1. The hull X of a tiling $\mathcal{L}$ is the compactification of $T \cong \{\sigma^t(\mathcal{L}) := \mathcal{L} + t : t \in T\}$ in the tiling metric: tilings are close if they agree on a large ball around 0 up to a small error, symbolic sequences are close if the agree on a large interval around 0.

2. The associated max. equicont. factor X_{max} is the compactification of T obtained from spectral theory: $\phi \in C(X)$ is an eigenfunction to eigenvalue k if

$$\sigma^t \phi(x) := \phi(x+t) = e^{ikt}\phi(x)$$

$X_{max} := X/\sim$ with $x \sim y$ if $\phi(x) = \phi(y)$ for all eigenfunctions.

The quotient map $\pi : X \to X_{max}$ is a factor map: it conjugates the actions σ^t.

(X, T) is determined by (X_{max}, T) and by what happens to the fibres $\pi^{-1}(z)$, $z \in X_{max}$.

Tilings/sequences whose fibres have finite cardinality are of particular interest.
The minimal cardinality is called the coincidence rank.

Three compactifications of T from an aperiodic tiling/sequence

3. The Ellis semigroup $E(X,T)$ of (X,T) is the compactification of $T \cong \{\sigma^t : X \to X | t \in T\}$ in $\mathcal{F}(X) = \{f : X \to X\}$ w. ptw. convergence

$$E(X,T) = \overline{\{\sigma^t | t \in T\}}^{\mathcal{F}(X)} \subset \mathcal{F}(X)$$

- Semigroup product is composition of functions
- The product map $(f,g) \mapsto fg$ is continuous in the left variable. Hence right multiplication is continuous: E is a compact right topological semigroup.

Topological orbits become algebraic orbits: for $x \in X$

$$\overline{\{\sigma^t(x) | t \in T\}}^{X} = E(X,T)(x)$$

The elements capture proximality: $x \sim_{prox} y$ if $\inf_{t\in T} \mathrm{dist}(\sigma^t(x), \sigma^t(y)) = 0$

$$x \sim_{prox} y \quad \text{iff} \quad \exists f \in E(X,T) : f(x) = f(y)$$

f can be taken to be idempotent: $ff = f$.

How can we characterise $E(X, T)$?

Ellis' idea was to study dynamical systems through their semigroups.

$E(X, T)$ has a lot of structure and can look pretty different for different dynamical systems. One can ask

- has it small or big cardinality? ($\leq$ continuum (tame ?) or $\geq 2^{cont.}$?)
- what are the topological properties of its space? (metrisable?)
- what are the topological properties of its elements? (continuous?)
- has its product stronger topological properties? (left-topological?, jointly continuous?)
- what are its algebraic properties? (group?, what is the ideal structure?, simple semigroup?)

Many of these questions have been investigated. I ask them for tiling systems where the answer to the brown questions is no and to the blue questions is interesting.

The three compactifications play together

If $\pi : (X, T) \to (Y, T)$ is a factor map (π surjective and intertwining the T-actions) then

$$\pi_* : E(X, T) \to E(Y, T), \; \pi_*(f)(y) = f(\pi^{-1}(x))$$

is a well-defined continuous semigroup morphism.

$(E(X, T), T)$ is a topol. dyn. system: $t \in T$ acts by left multiplication with σ^t.

If (Y, T) is equicontinuous and T abelian, then $(E(Y, T), T) \cong (Y, T)$.

The three compactifications play together:

$$E^{fib}(X, T) \hookrightarrow E(X, T) \overset{\pi_*}{\twoheadrightarrow} E(X_{max}, T) \cong (X_{max}, T)$$

$E^{fib}(X, T)$ are the functions of $E(X, T)$ which preserve the fibres $\pi^{-1}(z)$, $z \in X_{max}$.

- Study $E^{fib}(X, T)$ and X_{max} individually.
  If fibres are finite then $E^{fib}(X, T)$ is a topological semigroup!
- The extension problem is delicate, because $E(X, T)$ is only right topological.

Constant length substitutions

A substitution Θ, of constant length ℓ, on a finite alphabet $\mathcal{A}$, is a concatenation of ℓ maps $\theta_i : \mathcal{A} \to \mathcal{A}$,

$$\Theta(a) = \theta_0(a)\,\theta_1(a)\,\cdots\theta_{\ell-1}(a)$$

Θ extends to words $\Theta(ab) = \theta(a)\theta(b) = \theta_0(a)\cdots\theta_{\ell-1}(a)\theta_0(b)\cdots\theta_{\ell-1}(b)$ hence can be iterated. If all θ_i are bijections the subst. is called bijective.

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Example (Thue Morse): $\Theta = \text{id flip flip id}$

$$\begin{array}{ccccc} a & \mapsto & abba & \mapsto & abbabaabbaababba \\ b & \mapsto & baab & \mapsto & baababbaabbabaab \end{array}$$

A symbolic sequ. $(x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ is Θ-allowed if any part $x_k x_{k+1}\cdots x_{k+m}$ occurs in $\Theta^n(a)$ for some $n \in \mathbb{N}$ and $a \in \mathcal{A}$.

$X_\Theta \subset \mathcal{A}^{\mathbb{Z}}$ is the space of all Θ-allowed sequences. It is closed and shift invariant. $(X_\Theta, \mathbb{Z}, \sigma)$ is the dynamical system associated to Θ.

Assumptions:

- Θ is primitive (all letters occur in $\Theta(a)$),
- Θ is of standard form: θ_0 and $\theta_{\ell-1}$ are idempotents,
- the sequences are aperiodic (X_Θ is infinite)

There is an equicontinuous factor (the MEF if height is 1)

$$\begin{array}{ccc} X_\Theta & \stackrel{\sigma}{\to} & X_\Theta \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{Z}_\ell & \stackrel{+1}{\to} & \mathbb{Z}_\ell \end{array}$$

Here $\mathbb{Z}_\ell = \{0, \cdots, \ell-1\}^{\mathbb{N}}$ is the odometer (addition $+1$ in base ℓ with carry over).

There is $c \leq |\mathcal{A}|$ (coincidence or column rank) s.th. $|\pi^{-1}(z)| \geq c$ with equality for almost all $z \in \mathbb{Z}_\ell$. Call the latter points (and corresponding fibres) regular.

$$E^{fib}(X_\Theta, \mathbb{Z}) \hookrightarrow E(X_\Theta, \mathbb{Z}) \stackrel{\pi_*}{\twoheadrightarrow} E(\mathbb{Z}_\ell, \mathbb{Z}) \stackrel{ev_0}{\cong} \mathbb{Z}_\ell$$

$E^{fib}(X_\Theta, \mathbb{Z})$ are the fibre preserving elements of $E(X_\Theta, \mathbb{Z})$.

Aim: determine

$$E^{fib}(X_\Theta, \mathbb{Z}) \subset \prod_{z\in\mathbb{Z}_\ell} E^{fib}_z(X_\Theta, \mathbb{Z})$$

$E^{fib}_z(X_\Theta, \mathbb{Z})$ the restriction of $E^{fib}(X_\Theta, \mathbb{Z})$ to the fibre $\pi^{-1}(z)$.

$x, y \in X_\Theta$ are proximal iff $\exists$ idempotent $p \in E^{fib}_{\pi(x)}(X_\Theta, \mathbb{Z})$ s.th. $p(x) = y$

Thue Morse $\Theta = \mathrm{id}\ \mathrm{flip}\ \mathrm{flip}\ \mathrm{id}$

Regular fibres have $c = 2$ elements. Hence $E_z^{fib} \subset S_2$ for regular z.

There is only one orbit of singular fibres, that of $\pi^{-1}(0)$.
It consists of the 4 fixed points of Θ: $a.a$, $a.b$, $b.a$, b,b.

There are 4 proximal pairs, 2 forward $\begin{matrix} a.a \\ b.a \end{matrix}, \begin{matrix} b.b \\ a.b \end{matrix}$ and 2 backward $\begin{matrix} a.a \\ a.b \end{matrix}, \begin{matrix} b.b \\ b.a \end{matrix}$

Every proximal pair gives rise to 2 idempotents. Symmetry $a \leftrightarrow b$ relates these:

2 forward $\begin{matrix} a.a \\ b.a \\ b.b \\ a.b \end{matrix} \mapsto \begin{matrix} a.a \\ a.a \\ b.b \\ b.b \end{matrix}, \begin{matrix} a.a \\ b.a \\ b.b \\ a.b \end{matrix} \mapsto \begin{matrix} b.a \\ b.a \\ a.b \\ a.b \end{matrix}$, 2 backward $\begin{matrix} a.a \\ b.a \\ b.b \\ a.b \end{matrix} \mapsto \begin{matrix} a.a \\ b.b \\ b.b \\ a.a \end{matrix}, \begin{matrix} a.a \\ b.a \\ b.b \\ a.b \end{matrix} \mapsto \begin{matrix} a.b \\ b.a \\ b.a \\ a.b \end{matrix}$

$$E_0^{fib}(X_\Theta) = \{\mathrm{id}\} \sqcup \{1,2\} \times \overset{a}{S_2} \times \{\mathrm{forw.}, \mathrm{backw.}\}$$

with product twisted by a S_2-valued matrix a (Rees matrix semigroup)

$$(i, g, \epsilon) \star_a (i', g', \epsilon') = (i, g a_{\epsilon i'} g', \epsilon'), \quad a = \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & \mathrm{flip} \end{pmatrix}$$

Thue Morse $\Theta = \mathrm{id}\ \mathrm{flip}\ \mathrm{flip}\ \mathrm{id}$ and independence

$$E^{fib}(X_\Theta) \stackrel{?}{=} \prod_{z\in\mathbb{Z}_4} E^{fib}_z(X_\Theta)$$

i.e. if $f \in E^{fib}(X_\Theta)$ and $z, w \in \mathbb{Z}_4$, does $f|_{\pi^{-1}(z)}$ determine $f|_{\pi^{-1}(w)}$ or not?

Thue Morse $\Theta = \mathrm{id}\ \mathrm{flip}\ \mathrm{flip}\ \mathrm{id}$ and independence

$$E^{fib}(X_\Theta) \overset{?}{=} \prod_{z\in\mathbb{Z}_4} E_z^{fib}(X_\Theta)$$

i.e. if $f \in E^{fib}(X_\Theta)$ and $z, w \in \mathbb{Z}_4$, does $f|_{\pi^{-1}(z)}$ determine $f|_{\pi^{-1}(w)}$ or not?

We have total dependence along orbits $f|_{\pi^{-1}(z+1)} = \sigma f \sigma^{-1}\big|_{\pi^{-1}(z)}$.

Theorem (Yassawi, K. 2019)

Otherwise there is total independence, i.e.

$$E^{fib}(X_\Theta) \overset{!}{\cong} \prod_{[z]\in\mathbb{Z}_4/\mathbb{Z}} E_z^{fib}(X_\Theta), \quad E_z^{fib}(X_\Theta) = \mathbb{Z}_2, \text{ for } [z] \neq [0]$$

This holds for all bijective substitutions with some non-trivial $G_\Theta \subset S_{\mathcal{A}}$ (structure grp).

$$E_0^{fib}(X_\Theta) = \{\mathrm{id}\} \sqcup I \times \overset{a}{G_\Theta} \times \{\text{forw.}, \text{backw.}\}$$

This is the source of non-tameness of bijective substitutions.

Constant length substitutions $\Theta = \theta_0 \cdots \theta_{\ell-1}$

Differences with bijective substitutions

- The maps θ_i generate only a semigroup $\mathcal{S}_\Theta$.
- There may be Li-Yorke pairs (proximal pairs which are not asymptotic).
- There may be more than 1 orbit of singular fibres,
  in fact either finitely many, or uncountably many.

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- There may be more than 1 orbit of singular fibres, in fact either finitely many, or uncountably many.

How does this affect the calculation?

- $E_z^{fib}(X_\theta)$ is computable using the direct reading graph of the substitution, provided $z \in \mathbb{Z}_\ell \cong \{0, \cdots, \ell - 1\}^{\mathbb{N}}$ is eventually periodic. This must be the case if there are only finitely many orbits of singular fibres.
- If there is a Li-Yorke pair in fibre $\pi^{-1}(z)$ then $E_z^{fib}(X_\theta)$ is no longer nearly simple (the union of its kernel with $\{\mathrm{id}\}$).
- The question of independence is a lot harder.
- There may be other sources of non-tameness.

Li-Yorke pairs

A forward Li-Yorke pair is a forward proximal pair (x, y), i.e.

$$\inf_{t\in\mathbb{Z}^+} dist(\sigma^t(x), \sigma^t(y)) = 0$$

which is not forward asymptotic, i.e. it does not satisfy

$$\lim_{t\in\mathbb{Z}^+} dist(\sigma^t(x), \sigma^t(y)) = 0$$

Theorem (Blanchard etal. '07, Barge-K. '19)

$(X, \mathbb{Z})$ be a dynamical system with totally disconnected X. Are equivalent:

1. *Forward proximality agrees with forward asymptoticity (no forward Li-Yorke pairs)*
2. *$E(\mathbb{Z}^+)$ is the disjoint union of the acting semigroup group $\mathbb{Z}^+$ with its kernel*

Despite the presence of Li-Yorke pairs, $E_z^{fib}(X_\theta)$ is computable using the direct reading graph of the substitution, provided $z \in \mathbb{Z}_\ell \cong \{0, \cdots, \ell - 1\}^{\mathbb{N}}$ is eventually periodic. The latter is always the case if there are only finitely many orbits of singular fibres.

A substitution with Li-Yorke pair

Let Θ be the substitution on $\mathcal{A} = \{a, b, c\}$:

$$\begin{aligned} a &\mapsto aacaa \\ b &\mapsto abcaa \\ c &\mapsto accba \end{aligned}$$

Θ has a coincidence so $c = 1$. The only singular fibres are in the orbit of $\pi^{-1}(111\cdots)$ which contains exactly the fixed points of $\tilde{\Theta} := \sigma \circ \Theta$. There are three of them, in bijection to $\mathcal{A}$, obtained from the seeds a, b, c. We show two iterations:

$$\begin{array}{ccccc} .a & & a.acaa & & aacaaa.acaaaccbaaacaaaacaa \\ .b & \xrightarrow{\tilde{\theta}} & a.bcaa & \xrightarrow{\tilde{\theta}} & aacaaa.bcaaaccbaaacaaaacaa \\ .c & & a.ccba & & aacaaa.ccbaaccbaabcaaaacaa \end{array}$$

Proximality agrees with asymptot. backwards (to the left), but not forwards: seeds b and c give rise to a forward Li-Yorke pair. We find

$$E(X_\theta, \mathbb{Z}) = \{\sigma^n\}_{n\in\mathbb{Z}} \sqcup \{\phi^n\}_{n\in\mathbb{Z}} \sqcup \{p_a, p_b, p_c\} \times \mathbb{Z}_5$$

where $p_i(a_j) = a_i$ and $\phi(a) = a$, $\phi(b) = a$, $\phi(c) = b$. Multipliction is given by

$$p_i p_j = p_i, \quad \phi^2 = \pi_a, \quad \phi\pi_a = \pi_a, \ \ldots .$$

Independence $E^{fib}(X_\Theta) \overset{?}{\cong} \prod_{[z]\in\mathbb{Z}_\ell/\mathbb{Z}} E_z^{fib}(X_\Theta)$

Let $\Theta = \theta_0 \cdots \theta_{\ell-1}$ (primitive aperiodic), $\mathcal{S}_\Theta$ the semigroup generated by the θ_i. The column rank is

$$c = \min_{\theta\in S_\Theta} |\mathrm{im}\theta|$$

There is a group $G_\Theta \subset S_c$ s.th. $E_z^{fib}(X_\Theta) = G_\theta$ for regular z (structure group).

Theorem (2022)

$(X_\Theta, \mathbb{Z})$ factors onto a bijective substitution $(X_\eta, \mathbb{Z})$ over an alphabet with c letters with a sliding block code of radius 0 iff S_Θ has a unique minimal left ideal.

The factor map is $\mathcal{A} \to \mathcal{A}/\sim$ where $a \sim b$ if $\forall\theta \in S_\Theta$: $\theta(a) = \theta(b)$ and the bijective substitution is

$$\eta = \theta$$

If $c = 1$ then η is periodic. If η is aperiodic then the factor map is 1-1 on regular points.

In the above case, for any minimal idempotent $e \in \mathcal{E}^{fib}(X_\Theta)$,

$$eE^{fib}(X_\Theta)e \cong \prod_{[z]\in\mathbb{Z}_\ell/\mathbb{Z}} G_\Theta$$

Total independence along the group (non-tame).

Tameness and independence

A priori, elements of $E(X, T)$ are limits of nets. A Baire class 1 function is a ptw. limit of a sequence of continuous functions.

Definition

If all elements of $E(X, T)$ are Baire class 1 functions ($E(X, T)$ is the sequential compactification of the group action) then (X, T) (or $E(X, T)$) is called tame.[a]

[a] This is not the original formulation of tameness, which is due to Köhler

Definition

(X, T) contains an independence sequence if there is a pair of disjoint closed subsets $V_0, V_1 \subset X$ and an infinite subset $J \subset T$ such that for each function $\varphi : J \to \{0, 1\}$ there is $x \in X$ with

$$\sigma^j(x) \in V_{\varphi(j)} \qquad \forall j \in J.$$

Theorem (Köhler '95, Kerr-Li '07)

$E(T)$ is non-tame iff (X, T) contains an independence sequence.

Which systems are tame?

(Y,T) is equicontinuous if $\{\sigma^t | t \in T\}$ is an equicont. family.

(X,T) is almost automorphic if it admits a minimal equicontinuous factor $(X,T) \xrightarrow{\pi} (Y,T)$ which has a singleton fibre: $\exists y \in Y : |\pi^{-1}(y)| = 1$.

Theorem (Huang, Glasner, Fuhrmann, Jäger, Oertel 2018)

A minimal tame system with abelian T is almost automorphic and the set $\{y \in Y || \pi^{-1}(y)| = 1\}$ has full Haar measure. And it is uniquely ergodic.

A repetitive FLC Meyer set can only be tame if it is a regular model set.

Are regular model sets always tame?

Theorem (Aujogue 2011)

Almost canonical cut & project sets (regular euclidean model sets with "polyhedral windows") are tame.

Theorem (Fuhrmann-Kwietniak 2020)

There are non-tame regular model sets with internal space $\mathbb{T}$.

Other forms of non-tameness

Theorem (Fuhrmann-Yassawi-K. 2020)

A primitive aperiodic substitution of constant length with coincidence ($c = 1$) is tame iff there are only finitely many orbits of singular fibres.

This is a special case of a similar result for Toeplitz sequences [FYK 2020].

Other forms of non-tameness

Theorem (Fuhrmann-Yassawi-K. 2020)

A primitive aperiodic substitution of constant length with coincidence ($c = 1$) is tame iff there are only finitely many orbits of singular fibres.

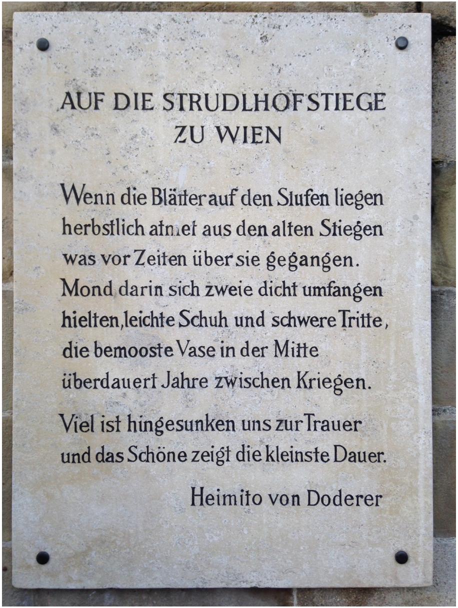
This is a special case of a similar result for Toeplitz sequences [FYK 2020].

There is an algorithm which determines the set of singular points of a primitive aperiodic substitution of constant length [Coven-Yassawi-Quas 2016].

Example:

$$\begin{array}{ccc} a & \mapsto & aabaa \\ b & \mapsto & abbaa \end{array} \quad \text{is tame,} \qquad \begin{array}{ccc} a & \mapsto & aaaba \\ b & \mapsto & abbaa \end{array} \quad \text{is wild}$$

In particular, strong orbit equivalence does not preserve tameness.



On the Strudlhof Steps in Vienna

When the leaves upon the steps are lying,
from the old stairs is heard an autumn sighing
of all that's gone across them in the past.
A moon in which a couple, holding fast,
embraces, lightweight shoes and heavy footfall,
the mossed urn in the middle, by the wall,
outlasts the years between the wars and dying.

So much is past and gone, to our dismay,
And beauty shows the frailest power to stay.