Understanding substitution tilings with pure point spectrum through a cut-and-project method

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Outline

Motivation

Question

The typical property of quasicrystal structures is that the diffraction pattern has a strong component of Bragg peaks which lacks periodic order.

1. What structures show the diffraction pattern consisting only of Bragg peaks?

2. Are their structures coming from projecting certain sections of high dimensional lattice structures?

Background

Background

1. [Hof ‘98] and [Schlottmann ‘00] has shown that a regular model set in $\mathbb{R}^d$ has pure point spectrum.
2. [Baake-Moody ‘04], [Baake-Lenz-Moody ‘07], [Strungaru ‘17] have discussed the relation between regular model sets and pure point spectrum.
3. [Dekking ‘78], [Lee-Moody ‘01], [Lee-Moody-Solomyak ‘03] have shown an equivalence between regular model sets and pure point spectrum in lattice substitution tilings on $\mathbb{R}^d$.
4. [Barge-Kwapisz ’06] has shown an equivalence between regular model sets and pure point spectrum in unimodular substitution tilings on $\mathbb{R}$.

Background

5. [Minervino-Thuswaldner '14] has shown an equivalence between regular model sets and pure point spectrum in substitution tilings on $\mathbb{R}$.

6. [Lee '07] has shown an equivalence between inter model sets and pure point spectrum in substitution tilings on $\mathbb{R}^d$.

7. [Lee-Akiyama-Lee '20] has shown an equivalence between regular model sets and pure point spectrum in unimodular substitution tilings on $\mathbb{R}^d$ with diagonalizable expansion map ϕ.

Question

Question

Can we say an equivalence between regular model sets and pure point spectrum in non-unimodular substitution tilings on $\mathbb{R}^d$ with diagonalizable expansion map ϕ?

Main Theorem

We assume that

1. $\mathcal{T}$ is a repetitive primitive substitution tiling on $\mathbb{R}^d$ with expansion map ϕ,
2. ϕ is diagonalizable,
3. all the eigenvalues of ϕ are algebraically conjugate with the same multiplicity,
4. $\mathcal{T}$ has a rigid structure.

Then $\mathcal{T}$ has pure point spectrum iff a control point set of $\mathcal{T}$ is a regular model set in a CPS with an internal space which is a product of a Euclidean space and a profinite group.

Preliminaries

Pure point spectrum

X: a collection of tilings made of tiles in $\mathcal{T}$,

$X_{\mathcal{T}} := \overline{\{x + \mathcal{T} : x \in \mathbb{R}^d\}}$ with a local topology on X.

$(X_{\mathcal{T}}, \mathbb{R}^d)$: $\mathbb{R}^d$-action by translations.

Assume $\exists$ unique invariant probability measure μ.

For $f \in L_2(X_{\mathcal{T}}, \mu)$, define $U_g f(\xi) = f(\xi - g)$ for $g \in \mathbb{R}^d$, $\xi \in X_{\mathcal{T}}$.

If $\exists\ \alpha \in \mathbb{R}^d$: $U_g f = e^{2\pi i \langle \alpha, g \rangle} f$ for all $g \in \mathbb{R}^d$, then f is an eigenfunction for the $\mathbb{R}^d$-action.

$\mathcal{T}$ is said to have pure point (dynamical) spectrum if the eigenfunctions span a dense subspace of $L_2(X_{\mathcal{T}}, \mu)$.

Cut-and-project scheme and regular model sets

A cut-and-project scheme (CPS) is a collection of spaces and mappings for which

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{K} & \xrightarrow{\pi_2} & \mathbb{K} \\ & & \cup & & \\ L & \longleftarrow & \widetilde{L} & \longrightarrow & L^* \\ x & \longleftarrow\!\shortmid & (x, x^*) & \longmapsto & x^* , \end{array} \tag{1}$$

where

(1) $\mathbb{K}$ is a locally compact Abelian group,

(2) $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times \mathbb{K}$,

(3) π_1 and π_2 are canonical projections such that $\pi_1|_{\widetilde{L}}$ is injective and $\pi_2(\widetilde{L})$ is dense in $\mathbb{K}$.

Here we denote by * the mapping $\pi_2 \cdot (\pi_1|_{\widetilde{L}})^{-1} : L \to \mathbb{K}$.

Regular model sets

$\curlywedge(V) := \{\pi_1(x) \in \mathbb{R}^d : x \in \widetilde{L}, \pi_2(x) \in V\}$, where $V \subset \mathbb{K}$.

$\Gamma \subset \mathbb{R}^d$ is a model set if $\Gamma = \curlywedge(W)$ where $W^\circ \neq \emptyset$ and $\overline{W}$ is compact in $\mathbb{K}$.

A model set Γ is regular if $\mu(\partial W) = 0$ with a Haar measure μ.

Substitution tilings

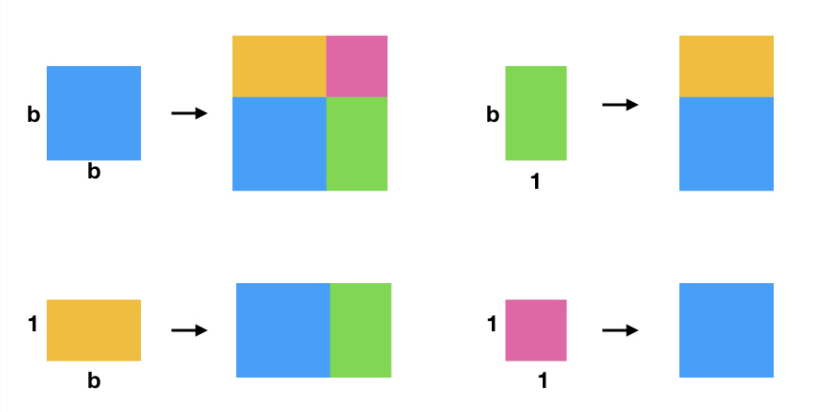


그림 1: 2-dimension Fibonacci substitution tiling (b is the golden ratio)

Substitution tilings

- Let $\mathcal{A} = \{T_1, \ldots, T_\kappa\}$ be a finite set of tiles in $\mathbb{R}^d$ with $T_i = (A_i, i)$ and $\mathcal{P}_\mathcal{A}$ be the set of patches made of tiles in $\mathcal{A}$.
- We say that $\omega : \mathcal{A} \to \mathcal{P}_\mathcal{A}$ is a substitution with expansion (linear) map ϕ if $\exists$ compact sets A_i's with $A_i = \overline{A_i^\circ} \neq \emptyset$ and finite sets $\mathcal{D}_{ij}$ such that

$$\omega(T_j) = \{u + T_i : u \in \mathcal{D}_{ij}, i \leq \kappa\}$$

with

$$\phi(A_j) = \dot{\bigcup}_{i \leq \kappa} (A_i + \mathcal{D}_{ij}), \quad \text{for } j \leq \kappa$$

where all sets in the right-hand side have disjoint interiors.

- When $\mathcal{T}$ is a tiling and $\omega(\mathcal{T}) = \mathcal{T}$, we say that $\mathcal{T}$ is a substitution tiling.
- We say that ϕ is unimodular if the minimal polynomial of ϕ over $\mathbb{Z}$ has constant term ± 1. Otherwise we say that ϕ is non-unimodular.

Rigid structure

For a tiling $\mathcal{T}$, define $\Xi(\mathcal{T}) := \{x \in \mathbb{R}^d : T = x + T', T, T' \in \mathcal{T}\}$.

Theorem (Kenyon '94, Solomyak '06)

Let $\mathcal{T}$ be a primitive substitution tiling in $\mathbb{R}^d$ with FLC for which $\phi(= \theta)$ is a similarity map for which $|\theta| > 1$. Then

$$\Xi(\mathcal{T}) \subset \mathbb{Z}[\theta]\alpha_1 + \cdots + \mathbb{Z}[\theta]\alpha_d$$

for some basis $\{\alpha_1, \cdots, \alpha_d\}$ in $\mathbb{R}^d$.

Rigid structure for $\mathcal{T}$

Theorem [Lee-Solomyak '12]

Let $\mathcal{T}$ be a primitive substitution tiling in $\mathbb{R}^d$ with an expansion map ϕ for which $\mathcal{T}$ has FLC. Assume that

(1) ϕ is diagonalizable over $\mathbb{C}$

(2) all the eigenvalues of ϕ are algebraically conjugate with the same multiplicity J.

Then $\exists$ an isomorphism $\rho : \mathbb{R}^d \to \mathbb{R}^d$ such that

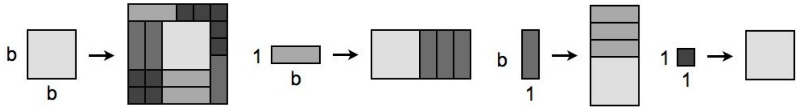
$$\rho\phi = \phi\rho \quad \text{and} \quad \Xi(\mathcal{T}) \subset \rho(\mathbb{Z}[\phi]\alpha_1 + \cdots + \mathbb{Z}[\phi]\alpha_J),$$

where m is the number of different eigenvalues of ϕ ($d = mJ$) and

$$(\alpha_j)_n = \begin{cases} 1 & \text{if } (j-1)m+1 \le n \le jm; \\ 0 & \text{else.} \end{cases}$$

(We call this rigid structure).

Frank-Robinson substitution tiling (Rigid structure)

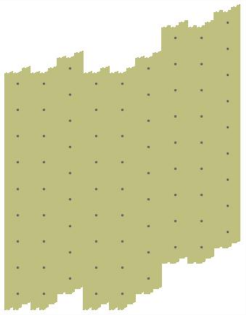


$$
\begin{aligned}
\phi A_1 &= (A_1 + (2,2)) \cup (A_2 + (2,0)) \cup (A_2 + (2,1)) \cup (A_2 + (0, b+2)) \\
&\quad \cup (A_3 + (0,2)) \cup (A_3 + (1,2)) \cup (A_3 + (b+2, 0)) \cup A_4 \cup (A_4 + (1,0)) \\
&\quad \cup (A_4 + (0,1)) \cup (A_4 + (1,1)) \cup (A_4 + (b+2, b)) \cup (A_4 + (b+2, b+1)) \\
&\quad \cup (A_4 + (b+2, b+2)) \cup (A_4 + (b+1, b+2)) \cup (A_4 + (b, b+2)) \\
\phi A_2 &= A_1 \cup (A_3 + (b,0)) \cup (A_3 + (b+1, 0)) \cup (A_3 + (b+2, 0)) \\
\phi A_3 &= A_1 \cup (A_2 + (0,b)) \cup (A_2 + (0, b+1)) \cup (A_2 + (0, b+2)) \\
\phi A_4 &= A_1 \,,
\end{aligned}
$$

where b is the largest root of $x^2 - x - 3 = 0$ (not a Pisot number) and $\phi = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$.

Note that $\Xi(\mathcal{T}) \subset \mathbb{Z}[\phi](1,0) + \mathbb{Z}[\phi](0,1)$. (Rigid structure)

Kenyon substitution tiling (Not rigid structure)



$\phi = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$: expansion map and

$\mathcal{D} = \{(0,-1),(0,0),(0,1),(-1,-1),(-1,0),(-1,1),(1,-1+a),(1,a),(1,1+a)\}$: digit set,

where $a \in \mathbb{R} \backslash \mathbb{Q}$.

Note that $\Xi(\mathcal{T}) \subset \mathbb{Z}[\phi](1,0) + \mathbb{Z}[\phi](0,1) + \mathbb{Z}[\phi](0,a)$.

(Not rigid structure)

Control point sets

Define a **tile map** $\gamma : \mathcal{T} \to \mathcal{T}$ such that $\forall\ T \in \mathcal{T}$, choose a tile γT on the patch $\omega(T)$; for all tiles of the same type, choose γT with the same relative position.

Define the **control point** for a tile $T \in \mathcal{T}$ by

$$\{c(T)\} = \bigcap_{m=0}^{\infty} \phi^{-m}(\gamma^m T). \tag{2}$$

The control points satisfy :

(a) $T' = T + c(T') - c(T)$, for any tiles T, T' of the same type;
(b) $\phi(c(T)) = c(\gamma T)$, for $T \in \mathcal{T}$.

Let $\mathcal{C} := \mathcal{C}(\mathcal{T}) = \{c(T) : T \in \mathcal{T}\}$ be a set of control points of the tiling $\mathcal{T}$ in $\mathbb{R}^d$.

Substitution tilings

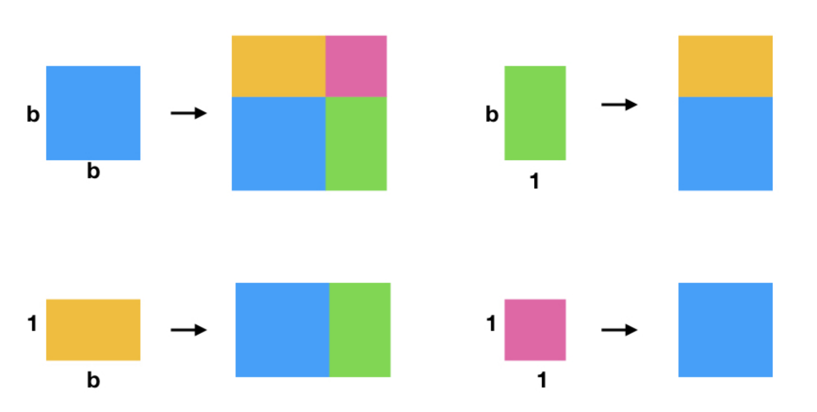


그림 2: 2-dimension Fibonacci substitution tiling (b is the golden ratio)

Control point sets

It is important to start with a control point set satisfying the following inclusion

$$\phi\langle\bigcup_{i\leq\kappa}\mathcal{C}_i\rangle_{\mathbb{Z}} \;\subset\; \langle\Xi(\mathcal{T})\rangle_{\mathbb{Z}}\,. \qquad (3)$$

For any given primitive substitution tilings, it is always possible to choose the control point set satisfying (3).

Control point sets

Consider a two letter substitution:

$$a \to aba \qquad b \to bab\,.$$

$$\cdots bababab|abababa \cdots\,.$$

Give a unit length interval for each letter, take the left end point of each tile and get a substitution point set

$$\Lambda_a = 2\mathbb{Z}, \Lambda_b = 1 + 2\mathbb{Z}, \Lambda_a \cup \Lambda_b = \mathbb{Z}.$$

Construct a CPS with an internal space which is a 3-adic completion $\overleftarrow{\mathbb{Z}_3}$ of $\mathbb{Z}$.

Since $x + 3^n\mathbb{Z} \nsubseteq 2\mathbb{Z}, \ \forall\, x \in \mathbb{Z}$ and $n \in \mathbb{N}$, Λ_a and Λ_b cannot be described as a model set using the internal space $\overleftarrow{\mathbb{Z}_3}$ for a CPS.

Note that the condition (3) is not satisfied.

Control point sets

However, take a tile map $\gamma : \mathcal{T} \to \mathcal{T}$ for which

$$\gamma(T) = 3x + T_a \quad \text{and} \quad \gamma(S) = 3y + 1 + T_a,$$

where $T = x + T_a$, $S = y + T_b \in \mathcal{T}$, $x \in 2\mathbb{Z}$, and $y \in 1 + 2\mathbb{Z}$.

The control point set $\mathcal{C}(\mathcal{T}) = (\mathcal{C}_i)_{i \in \{a,b\}}$ is $\mathcal{C}_a = 2\mathbb{Z}$ and $\mathcal{C}_b = \frac{4}{3} + 2\mathbb{Z}$. Let

$$L := \langle \mathcal{C}_a, \mathcal{C}_b \rangle_{\mathbb{Z}} = \frac{2\mathbb{Z}}{3}.$$

Note that $3L \subset \Xi(\mathcal{T})(= 2\mathbb{Z})$ which satisfys the condition (3).

Control point sets

Consider 3-adic completion $\overleftarrow{L}_3$ of L

$$\begin{aligned}\overleftarrow{L}_3 &:= \lim_{\leftarrow k} L/3^k L \\ &= \left(\frac{2\mathbb{Z}}{3}\right)/2\mathbb{Z} \leftarrow \left(\frac{2\mathbb{Z}}{3}\right)/3\cdot 2\mathbb{Z} \leftarrow \left(\frac{2\mathbb{Z}}{3}\right)/3^2\cdot 2\mathbb{Z} \leftarrow \cdots .\end{aligned}$$

So

$$\mathcal{C}_a = 2\mathbb{Z} = \curlywedge(3\cdot\overleftarrow{L}_3), \quad \mathcal{C}_b = \tfrac{4}{3} + 2\mathbb{Z} = \curlywedge(\tfrac{4}{3} + 3\cdot\overleftarrow{L}_3) .$$

Therefore $\mathcal{C}(\mathcal{T})$ can be described as a model set.

Main result

Main Theorem

We assume that

1. $\mathcal{T}$ is a repetitive primitive substitution tiling on $\mathbb{R}^d$ with expansion map ϕ,
2. ϕ is diagonalizable,
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Then $\mathcal{T}$ has pure point spectrum iff a control point set of $\mathcal{T}$ is a regular model set in CPS with an internal space which is a product of a Euclidean space and a profinite group.

Example (Construction of a CPS)

Example

Consider a substitution:

$$a \to aab \qquad b \to abab\,.$$

The substitution matrix

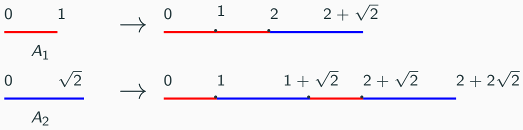
$$\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$$

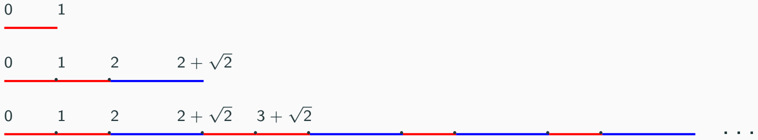
has the Perron-Frobenius eigenvalue $\lambda := 2 + \sqrt{2}$ and $x^2 - 4x + 2$ is the minimal polynomial of λ over $\mathbb{Z}$.

The substitution is non-unimodular.

Example

$$\lambda A_1 = A_1 \cup (A_1 + 1) \cup (A_2 + 2)$$
$$\lambda A_2 = A_1 \cup (A_2 + 1) \cup (A_1 + 1 + \sqrt{2}) \cup (A_2 + 2 + \sqrt{2})$$





Example

[Baake-Moody-Schlottmann ‘98] has shown that this two letter substitution tiling has pure point spectrum.

One can also check the pure point spectrum by algorithmic computation given in [Akiyama-Lee ‘11].

Example (Construction of a CPS)

Let $\mathcal{C}(\mathcal{T})$ be a representative point set of $\mathcal{T}$.

It is possible to take $\mathcal{C}(\mathcal{T})$ satisfying

$$\mathcal{C}(\mathcal{T}) \subset \mathbb{Z}[\lambda] = \{a + b\lambda \mid a, b \in \mathbb{Z}\} =: L.$$

Since

$$\begin{array}{rcl} \lambda(1) & = & 0 \cdot 1 + 1 \cdot \lambda \\ \lambda(\lambda) & = & \lambda^2 = 4\lambda - 2 = -2 \cdot 1 + 4 \cdot \lambda \,, \end{array}$$

we get

$$M = \begin{pmatrix} 0 & -2 \\ 1 & 4 \end{pmatrix} .$$

Example (Construction of a CPS)

We identify L with a lattice $\mathbb{Z}^2$ on $\mathbb{R}^2$ by the folllowing map

$$\pi : L \to \mathbb{Z}^2, \quad \pi(c_1 + c_2\lambda) = (c_1, c_2), \tag{4}$$

where $L = \mathbb{Z}[\lambda]$ and $c_1, c_2 \in \mathbb{Z}$. Then

$$\pi(\lambda \mathbf{v}) = M\pi(\mathbf{v}), \quad \forall\, \mathbf{v} \in L.$$

Since it is non-unimodular case, $M\mathbb{Z}^2 \subsetneq \mathbb{Z}^2$.

Thus the following M-adic completion of $\mathbb{Z}^2$ is not trivial.

$$\begin{aligned}
\overleftarrow{\mathbb{Z}^2_M} &:= \lim_{\leftarrow k} \mathbb{Z}^2/M^k\mathbb{Z}^2 \qquad (5)\\
&= \mathbb{Z}^2/M\mathbb{Z}^2 \leftarrow \mathbb{Z}^2/M^2\mathbb{Z}^2 \leftarrow \mathbb{Z}^2/M^3\mathbb{Z}^2 \leftarrow \cdots \\
&= \{(x_1 + M\mathbb{Z}^2, x_2 + M^2\mathbb{Z}^2, x_3 + M^3\mathbb{Z}^2, \dots) \mid \\
&\qquad x_1 \in \mathbb{Z}^2, x_k \in x_{k-1} + M^{k-1}\mathbb{Z}^2 \quad \text{for each integer } k \geq 2\}.
\end{aligned}$$

Example (Construction of a CPS)

Note that $\overleftarrow{\mathbb{Z}^2_M}$ contains a canonical copy of $\mathbb{Z}^2$ via the mapping

$\imath : \mathbb{Z}^2 \to \overleftarrow{\mathbb{Z}^2_M}$ such that $x \mapsto (x + M\mathbb{Z}^2, x + M^2\mathbb{Z}^2, x + M^3\mathbb{Z}^2, \dots)$.

We identify $\mathbb{Z}^2$ with its image in $\overleftarrow{\mathbb{Z}^2_M}$.

Example (Construction of a CPS)

Let

$$\begin{array}{rcl} \Psi \ : \ \mathbb{Z}[\lambda] & \to & \mathbb{R} \times \overleftarrow{\mathbb{Z}_M^2} \\ P(\lambda) & \mapsto & (P(\overline{\lambda}), (\iota \circ \pi)(P(\lambda))), \end{array} \qquad (6)$$

where $P(x) \in \mathbb{Z}[x]$, $\overline{\lambda} = 2 - \sqrt{2}$.

Example (Construction of a CPS)

Consider a cut-and-project scheme(CPS):

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R} \times (\mathbb{R} \times \overleftarrow{\mathbb{Z}_M^2}) & \xrightarrow{\pi_2} & \mathbb{R} \times \overleftarrow{\mathbb{Z}_M^2} \\ & & \cup & & \\ L & \longleftarrow & \widetilde{L} & \longrightarrow & \Psi(L) \\ & & \Cup & & \\ x & \longleftarrow\!\!\!\shortmid & (x, \Psi(x)) & \longmapsto & \Psi(x)\,, \end{array} \qquad (7)$$

where π_1 and π_2 are canonical projections, and $\widetilde{L} = \{(x, \Psi(x)) : x \in L\}$.

Example (Construction of a CPS)

Lemma

1. $\pi_1|_{\widetilde{L}}$ is injective.
2. $\pi_2(\widetilde{L})(= \Psi(L))$ is dense in $\mathbb{R} \times \overleftarrow{\mathbb{Z}^2_M}$.
3. $\widetilde{L}$ is a lattice in $\mathbb{R} \times (\mathbb{R} \times \overleftarrow{\mathbb{Z}^2_M})$.

THANK YOU!