When is a locally convex space Eberlein-Grothendieck?

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July 18-22, 2022

36th Summer Topology Conference, VIENNA

This is a joint work with Jerzy Kąkol.
The presentation is based on the results from

Jerzy Kąkol, Arkady Leiderman,
When is a locally convex space Eberlein-Grothendieck? (2022),
https://arxiv.org/abs/2206.10684

The abbreviation LCS means locally convex space.
All topological spaces are assumed to be Tychonoff and all vector spaces are over the field of real numbers $\mathbb{R}$.

By $C_k(X)$ and $C_p(X)$ we mean the space $C(X)$ of real-valued continuous functions defined on a Tychonoff space X equipped with the compact-open and pointwise convergence topology, respectively.

For a LCS E we denote by w the weak topology $w = \sigma(E, E^*)$ of E. The w^*-topology of the dual E^* is denoted by w^*.

Eberlein-Grothendieck topological space

The following significant statement which is intentionally formulated below in a simplified form is due to A. Grothendieck.

Theorem 0.0.

Let X be a compact space, and let A be a countably compact set in $C_p(X)$. Then the closure of A in $C_p(X)$ is compact.

Definition 0.1.

Following to A. V. Arkhangel'skii, a topological space Y is called an *Eberlein-Grothendieck space*, if there exists a homeomorphic embedding of Y into the space $C_p(K)$ for some compact space K.

1. A compact space is Eberlein-Grothendieck iff it is an Eberlein compact.
2. Every metrizable space is Eberlein-Grothendieck.
3. Every Eberlein-Grothendieck topological space Y has countable *tightness*.

For a normed space E there exists even a canonical *linear* embedding $T : (E, w) \to C_p(K)$, where K is the closed unit ball in E^* endowed with the weak*-topology w^*. Being motivated by this simple observation we introduce the following

Definition 0.2.

For a LCS E the space (E, w) is called a *linearly Eberlein-Grothendieck space* if (E, w) can be linearly embedded into $C_p(K)$ for some compact space K.

In our work we undertake a systematic study of those lcs E such that (E, w) is Eberlein-Grothendieck / linearly Eberlein-Grothendieck space.

Both results below are due to O. Okunev.

Theorem 0.3.

If a LCS H is a continuous open image of a subspace of $C_p(K)$ for some compact space K, then the dual lcs (H^*, w^*) is σ-compact.

Corollary 0.4.

A LCS $C_p(X)$ is Eberlein-Grothendieck if and only if X is σ-compact.

Proposition 1.1.

For every LCS E the following are equivalent

(a) (E, w) is Eberlein-Grothendieck.

(b) The dual lcs (E^*, w^*) is σ-compact.

Proposition 1.2.

For every metrizable LCS E, the space (E, w) is Eberlein-Grothendieck.

The class of LCS E with Eberlein-Grothendieck (E, w) is invariant under certain basic topological operations.

Proposition 1.3.

(a) Let (E, w) be an Eberlein-Grothendieck LCS. If there is a linear continuous quotient mapping π from E onto a LCS F, then (F, w) is also Eberlein-Grothendieck.

(b) Let (E, w) be an Eberlein-Grothendieck LCS. Then (F, w) is Eberlein-Grothendieck for every linear subspace $F \subset E$.

(c) Let (E_n, w) be an Eberlein-Grothendieck LCS, where $n \in \omega$. Then the countable product $E = \prod_{n\in\omega} E_n$ also has the property that (E, w) is Eberlein-Grothendieck.

Remark 1.4.

Inductive limit of the countable sequence of lcs $(E_n)_n$, where each (E_n, w) is Eberlein-Grothendieck, does not have to satisfy the same property. Denote by φ the $\aleph_0$-dimensional vector space endowed with the finest locally convex topology. The space φ can be identified with the strict inductive limit of the sequence of Euclidean spaces $\mathbb{R}^n$, but (φ, w) is not Eberlein-Grothendieck.

For the brevity let us denote by EG_k the class of spaces X such that a LCS $(C_k(X), w)$ is Eberlein-Grothendieck.

Theorem 2.1.

If $X \in EG_k$, then X is σ-compact.

Proof

We observe that X is homeomorphic to a closed subspace of $(C_k(X)^*, w^*)$.

Recall that a space X is said to be *hemicompact* if there is a sequence $\{K_n : n \in \omega\}$ of compact subsets of X with the following property: if $K \subset X$ is compact then $K \subset K_n$ for some $n \in \omega$.

It is known that $C_k(X)$ is metrizable if and only if X is hemicompact.
So, if X is hemicompact, then $X \in EG_k$.

Main Problem

Is the converse true, i.e. are the following properties equivalent:
(1) $(C_k(X), w)$ is an Eberlein-Grothendieck lcs
and
(2) X is hemicompact?

We show that the answer to that Main Problem is positive for all first-countable spaces X.

Theorem 2.2.

Let X be a first-countable space. The following are equivalent

(a) X is hemicompact.

(b) X is both σ-compact and locally compact.

(c) $C_k(X)$ is metrizable.

(d) $X \in EG_k$, i.e. $(C_k(X), w)$ is Eberlein-Grothendieck.

Idea of the proof

In order to prove (d) $\implies$ (b) it suffices to show that $X \in EG_k$ implies that X is locally compact. If $X \in EG_k$ then X is σ-compact, hence X is paracompact. A first-countable paracompact space is locally compact if and only if it does not contain a closed subspace homeomorphic to the *metric fan* M.

Metric fan

The *metric fan* M is a metrizable space defined as follows. As a set, M is the union of countably many disjoint countable sequences $M_i = \{x_{in} : n \in \mathbb{N}\}, i \in \mathbb{N}$ plus a point p "at infinity"; all points besides p are isolated in M, and a basic neighborhood U_n of p consists of p and all points from M_i such that $n \leq i$. Thus, the metric fan M can be represented as a countable union of disjoint closed discrete layers M_i, and a single non-isolated point p such that for every choice $y_i \in M_i$ the sequence $(y_i)_i$ converges to p in M.

Idea of the proof

On the contrary, assume that X is not locally compact. Then X contains a closed copy of M and consequently $M \in EG_k$. This is equivalent to the claim that $(C_k(M)^*, w^*)$ is σ-compact. Then we show that the opposite is true: $(C_k(M)^*, w^*)$ is not σ-compact because it contains a closed copy of the space of irrationals $\mathbb{J} \cong \mathbb{N}^{\mathbb{N}}$, which is not σ-compact.

Theorem 2.3.

Let X be a metrizable space. The following are equivalent

(a) X is hemicompact.

(b) Either X is compact, or there is a metrizable compact K such that X is homeomorphic to $K \setminus \{p\}$, where p is a non-isolated point of K.

(c) $C_k(X)$ is metrizable.

(d) $X \in EG_k$, i.e. $(C_k(X), w)$ is Eberlein-Grothendieck.

3. Linearly Eberlein-Grothendieck LCS

Definition 3.1.

By analogy with topological groups, a topological vector space L is called *compactly generated* if L has a compact basis K, meaning that the linear span of K is equal to L.

We will show that compactly generated LCS play an important role in our study.

Theorem 3.2.

For every E, a LCS (E, w) is linearly Eberlein-Grothendieck if and only if the dual LCS (E^*, w^*) is compactly generated.

A closed subgroup of an arbitrary compactly generated abelian group in general does not have to be compactly generated. Even a compactly generated free topological vector space $V[0,1]$ contains a closed linear subspace which is not compactly generated. However, for LCS we have

Theorem 3.3.

Every closed linear subspace of a compactly generated LCS is also compactly generated.

Theorem 3.4.

Let (E, w) be a linearly Eberlein-Grothendieck LCS. If there is a linear continuous quotient mapping π from E onto a LCS F, then (F, w) is also linearly Eberlein-Grothendieck.

Proposition 3.5.

Let E be a Fréchet space. Then (E, w) is linearly Eberlein-Grothendieck if and only if the metric of E can be generated by a complete norm, i.e. E is isomorphic to a Banach space.

A LCS that is a locally convex inductive limit of a countable inductive system of Fréchet spaces is called a *(LF)-space*.

Proposition 3.6.

Let E be an (LF)-space. Then (E, w) is linearly Eberlein-Grothendieck if and only if E is normable.

Theorem 3.7.

For any Tychonoff space X the following conditions are equivalent:

(a) X is compact;

(b) $C_p(X)$ is linearly Eberlein-Grothendieck;

(c) $(C_k(X), w)$ is linearly Eberlein-Grothendieck.

Idea of the proof

Assuming (b) we deduce that X is σ-compact. On the other hand, X must be pseudocompact because otherwise $C_p(X)$ would contain an isomorphic copy of $\mathbb{R}^\omega$, which is not linearly Eberlein-Grothendieck. Every σ-compact and pseudocompact space is compact and the proof of (a) is finished.
As before, assuming (c) we deduce that X is σ-compact. Similarly to the earlier case, X must be pseudocompact because otherwise $C_k(X)$ would contain an isomorphic copy of $\mathbb{R}^\omega$.

Example 3.8.

If $\Omega \subset \mathbb{R}^n$ is an open set, then the space of test functions $\mathfrak{D}(\Omega)$ is a complete Montel (LF)-space. As usual, $\mathfrak{D}'(\Omega)$ denotes its strong dual, the space of distributions. Not $\mathfrak{D}(\Omega)$ nor $\mathfrak{D}'(\Omega)$ is metrizable, they are not even sequential. One can show that (E, w) is not Eberlein-Grothendieck for E both $\mathfrak{D}(\Omega)$ and $\mathfrak{D}'(\Omega)$. Also both $\mathfrak{D}(\Omega)$ and $\mathfrak{D}'(\Omega)$ considered with the original topologies are not Eberlein-Grothendieck.

Example 3.9.

Let E be the lcs $l_\infty = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sup_n |x_n| < \infty\}$ equipped with the topology of pointwise convergence. If we consider the canonical linear mapping π of restriction from $C_p(\beta\mathbb{N})$ into $\mathbb{R}^{\mathbb{N}}$, then the image of π is exactly E. Since the mapping π is not quotient, by this way we cannot decide whether E is linearly Eberlein-Grothendieck. However, it has been proved that the space $C_p(\beta\mathbb{N})$ admits (another) linear continuous and quotient mapping onto E. Hence E is linearly Eberlein-Grothendieck.

Thank you !