K-theory of two-dimensional substitution tiling spaces from AF-algebras

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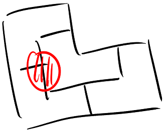
Tiling Spaces

A *tiling* T in $\mathbb{R}^2$ is a partition into sets of finite area, called *tiles*.
A prototile is an equivalence class of tiles, up to translation.
The associated *tiling space* Ω_T is $\overline{\{T - v : v \in \mathbb{R}^2\}}^d$ under an appropriate metric.
The topology is generated by cylinder sets of the form of patches.



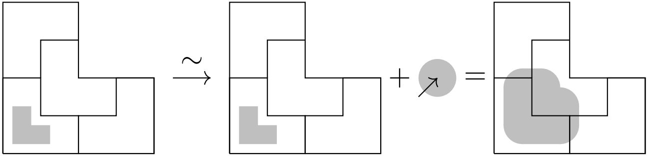
Translations

There is an action $\mathbb{R}^2 \circlearrowright \Omega_T$ with $T \mapsto T - v$.

Definition

This action gives a topological groupoid, the *unstable groupoid*, $G^u = \{(T', T' - v) \in \Omega_T^2\}$.

Basic open sets: $(\text{cylinder}, \text{cylinder} - B_r(v)) = B_r(v) \times \text{cylinder}$.



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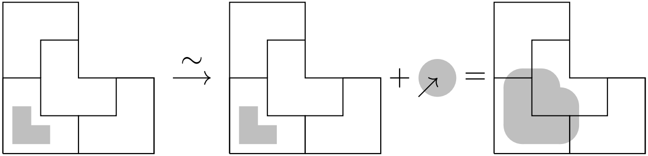

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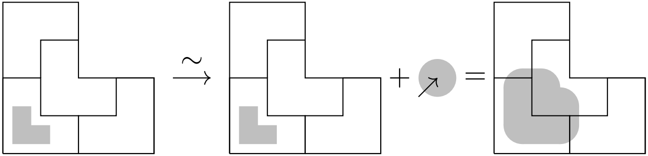

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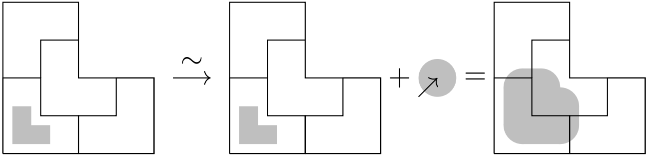
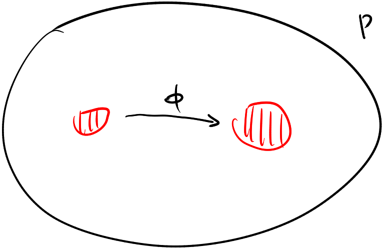


Partial Homeomorphisms

Fact

Same as the topology generated by partial homeomorphisms[1], *or homeomorphisms that are only defined between cylinder sets.*



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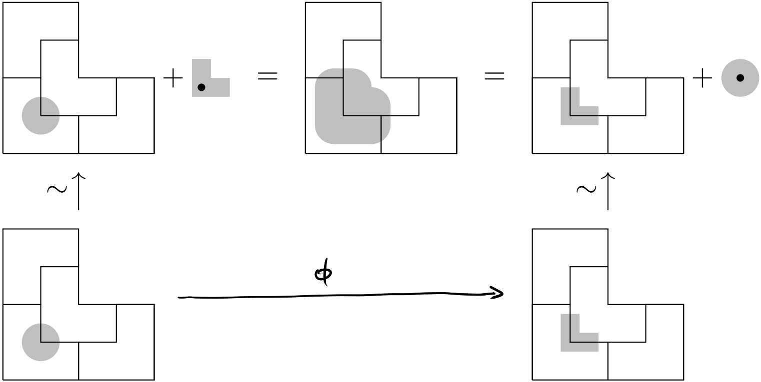

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[1] "Partially-defined homeomorphisms."

Operator K-theory

“Definition”

$K_0(G^u)$ is generated by the collection of cylinder sets of Ω_T, up to partial homeomorphisms.

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[2] Details only work on punctures.

Operator K-theory

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Geometry: the collection of coordinate charts, up to coordinate transformations.

Operator algebraic[2] terminology:

- (Basic "rank-1") projections ⇔ cylinder sets ⇔ coordinate charts, and
- Partial isometries ⇔ basic open sets of G^u/partial homeomorphisms ⇔ coordinate transformations.

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$K_1(G^u)$ . . . ? Harder to define, related to the action.
Will deduce using exactness.

---

[2] Details only work on punctures.

Operator K-theory

- Connes-Thom and Chern isomorphisms: rationally, K-theory is isomorphic to direct sum of Čech cohomology groups of the same parity (up to parity of dimension).
- Anderson-Putnam '98: dimensions 1/2 true without $\otimes\mathbb{Q}$.
- Forrest-Hunton '99: if torsion-free, true without $\otimes\mathbb{Q}$.

Relative K-theory

$$K_0(G) = \{\underbrace{\text{cylinder}}_{\text{same}}\} / \{\text{partial } \underbrace{\text{homeo}}_{\text{smaller}}\}$$

Origins: Karoubi's book (topology), Haslehurst (operator algebras '21)

Suppose that $\imath : G \hookrightarrow G^u$ (open subgroupoid).

"Definition"

$K_0(G; G^u)$ is generated by the collection of partial homeomorphisms in G^u, up to partial homeomorphisms in G.

Relative K-theory

$K_0(G)/K_0(G^u)$

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Suppose that $\iota : G \hookrightarrow G^u$ (open subgroupoid).

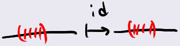
"Definition"

$K_0(G; G^u)$ is generated by the collection of partial homeomorphisms in G^u, up to partial homeomorphisms in G.

Relative K-theory

Theorem (Haslehurst '21)

The 6-term sequence



$$\begin{tikzcd}
K_0(G;G^u) \arrow[r, "\mathrm{ev}"] & K_0(G) \arrow[r, "\iota_*"] & K_0(G^u) \arrow[d] \\
K_1(G^u) \arrow[u] & K_1(G) \arrow[l] & K_1(G;G^u) \arrow[l]
\end{tikzcd}$$

is exact, where ev *sends a partial homeomorphism* $\text{cylinder}_1 \to \text{cylinder}_2$ *to the formal difference* $[\text{cylinder}_1] - [\text{cylinder}_2]$.

Remark

We want to pick $G \leq G^u$ *correctly so that no nontrivial action exists*[3].

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[3] Orbit-breaking.

Relative K-theory

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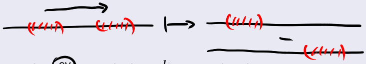

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We want to pick $G \leq G^u$ correctly so that no nontrivial action exists[3].

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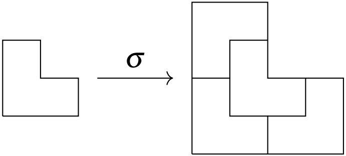
[3] Orbit-breaking.

Substitutions

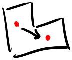
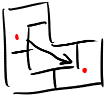
Ω_T arises from a *substitution* if there exists σ from the set of tiles to the set of patches that

1. Expands a tile by some $\lambda > 1$ and
2. Subdivides it into tiles.



$\sigma^n(t)$ is called a *level-n supertile*.

Operator K-Theory

$AF_0 =$  $AF_1 =$ 

Definition

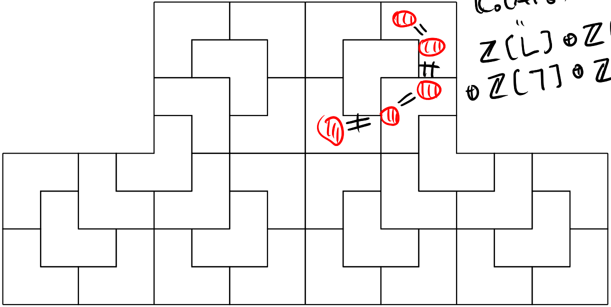
The *approximately finite-dimensional (AF) groupoid* is the subgroupoid of G^u where the pairs of tilings have both of their origins belonging to the same (sufficiently-high level) supertile.

By taking AF_n to be pairs of tilings whose origins belong to the same level-n supertile, we get $AF = \varinjlim_n (AF_n, \sigma_*)$.

$AF_0 \leq AF_1 \leq \cdots$

$\rightsquigarrow AF = \bigcup_n AF_n$

Operator *K*-Theory



$K_0(AF_0)$
$\parallel$
$\mathbb{Z}[\llcorner] \oplus \mathbb{Z}[\lrcorner]$
$\oplus \mathbb{Z}[\urcorner] \oplus \mathbb{Z}[\ulcorner]$

$K_0(AF)$[4] $= \varinjlim_n (K_0(AF_n), \sigma^\top)$, and

$K_0(AF_n) = \mathbb{Z}^{\#\text{level-}n\text{ supertiles}}$.

The AF_n-groupoid has no nontrivial action, i.e. $K_1(AF_n) = 0$.

Therefore $K_1(AF) = 0$.

$K_0(AF; G^u) = \varinjlim_n (K_0(AF_n; G^u), \sigma^\top)$.

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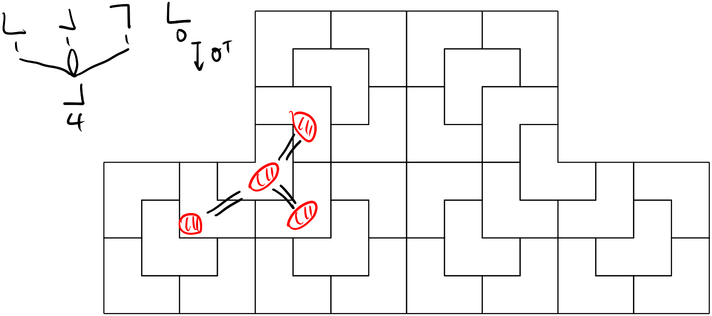
[4] Dimension group of the Bratteli diagram associated to $\sigma^\top$.

Operator K-Theory



$K_0(AF)^4 = \varinjlim_n (K_0(AF_n), \sigma^\top)$, and

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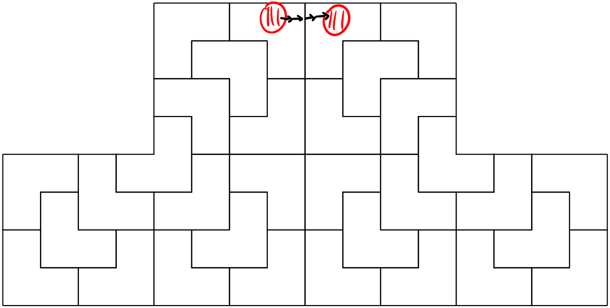

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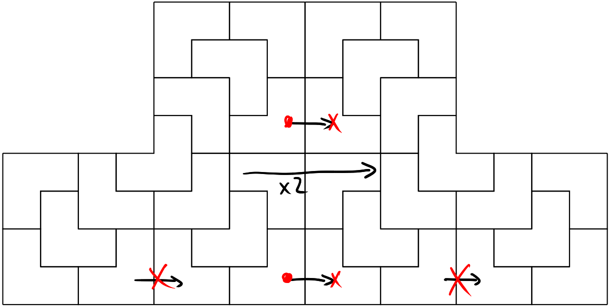

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[4] Dimension group of the Bratteli diagram associated to $\sigma^\top$.

Operator K-theory

Assume aperiodicity, primitivity, finite local complexity, and collaring.

- Putnam '89, Kellendonk '95: in dimension 1, the AF-groupoid is "large enough" to reconstruct the K-theory.
- Julien-Savinien '16: this works for the square version of the chair tiling if one uses both AF- and $AF^{(1)}$-groupoids[5].

We will apply the six-term sequence in relative K-theory to the inclusion $\iota : AF \hookrightarrow G^u$ to show this holds for all dimension 2 substitution tiling spaces.

---

[5] This does *not* have the same K-groups as relative K-groups, except in specific situations.

K-theory–Čech cohomology isomorphism

Theorem (L.)

For $d = 1$,

$$\begin{tikzcd}
 & \text{ker} & & & \text{cochain complex} & \text{coker} & \\
0 \arrow[r] & \check{H}^0(\Omega_T) \arrow[r, hook] \arrow[d] & \varinjlim_n(C_n^0,\sigma^\top) \arrow[r, "\delta^0"] \arrow[d, "\mathrm{th}", "\wr"'] & \varinjlim_n(C_n^1,\sigma^\top) \arrow[r, two heads] \arrow[d, equal] \arrow[dl, phantom, "2"] & & \check{H}^1(\Omega_T) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1(G^u) \arrow[r, hook] & K_0(AF;G^u) \arrow[r, "\mathrm{ev}"] & K_0(AF) \arrow[rr, two heads, "\iota_*"] & & K_0(G^u) \arrow[r] & 0
\end{tikzcd}$$

and for $d = 2$,

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^1(\Omega_T) \arrow[r, hook] \arrow[ddd] & \varinjlim_n(\frac{C_n^1}{\mathrm{im}\delta_n^0},\widetilde{\sigma^\top}) \arrow[r, "\widetilde{\delta^1}"] \arrow[ddd, "\widetilde{\mathrm{th}}"] & \varinjlim_n(C_n^2,\sigma^\top) \arrow[r, two heads] & \check{H}^2(\Omega_T) \arrow[r] & 0 & \\
 & & & \oplus & \oplus & \oplus & \\
 & & & 0 \arrow[r] \arrow[d, equal] & \check{H}^0(\Omega_T) \arrow[r, hook, two heads] \arrow[d] & \check{H}^0(\Omega_T) \arrow[r] \arrow[d] & 0 \\
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are isomorphisms of exact sequences.

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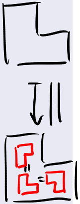
 & & & \varinjlim (K_0(AF_n), \sigma^\top) & &
\end{tikzcd}$$



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 & & & \oplus & \oplus & \oplus & \\
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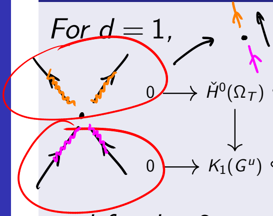

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 & & & \oplus & \oplus & \oplus & \\
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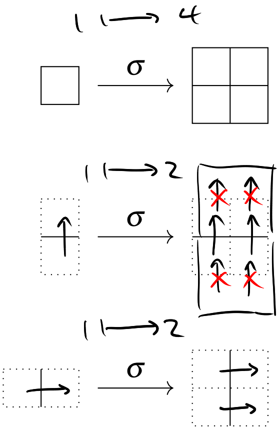
are isomorphisms of exact sequences.

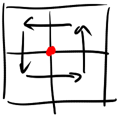
Square substitution



$K_0(AF) = \mathbb{Z}[1/4]$

$K_0(AF; G^u) = \langle \rightarrow, \uparrow \rangle^{\times 2} / \cancel{\circlearrowleft}$

$\quad = \mathbb{Z}[1/2]^2$



$\rightarrow = -\leftarrow$

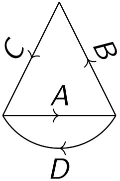
$\uparrow = -\downarrow$

$ev(\rightarrow) = \square - \square = 0$

$$\begin{tikzcd}
\mathbb{Z}[1/2]^2 \arrow[r, "0"] & \mathbb{Z}[1/4] \arrow[r, "\oplus"] & K_0(G^u) \arrow[d, "\oplus"] \\
K_1(G^u) \arrow[u, "\wr"] & 0 \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

Interpretation

$$\begin{aligned}
A &\xmapsto{\sigma^2} (AD)AB(C)\\
B &\xmapsto{\sigma^2} (B)CAB(C)\\
C &\xmapsto{\sigma^2} (AB)CAD(A)\\
D &\xmapsto{\sigma^2} (B)CAD(A)
\end{aligned}$$



Interpretation

$$A \xmapsto{\sigma^2} (AD)AB(C)$$
$$B \xmapsto{\sigma^2} (B)CAB(C)$$
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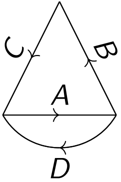

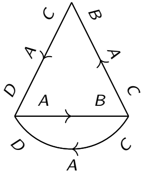

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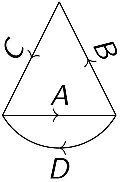

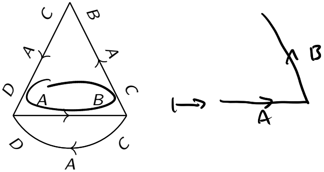

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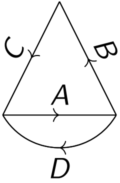

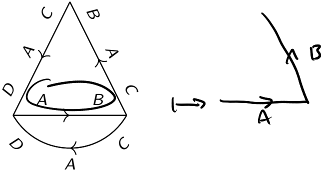

Then the substitution rule turns into an “evaluation” map.

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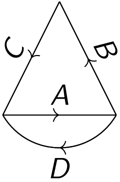
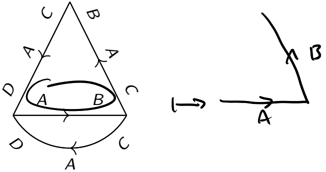




Then the substitution rule turns into an "evaluation" map. Applying this to $\widetilde{\delta_n^1} : C_n^1/\mathrm{im}\delta_n^0 \to C_n^2$ turns the coboundary map into an evaluation map, resulting in $\mathrm{ev} : K_0(AF; G^u) \to K_0(AF)$.

Applications[7]

Relative 6-term sequence

$$\begin{tikzcd}
K_0(AF; G^u) \arrow[r, "\mathrm{ev}"] & K_0(AF) \arrow[r, "\iota_*"] & K_0(G^u) \arrow[d] \\
K_1(G^u) \arrow[u] & K_1(AF) \arrow[l] & K_1(AF; G^u) \arrow[l]
\end{tikzcd}$$

Half-hex

$$\begin{tikzcd}
\mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^2 \arrow[r, "\mathrm{ev}"] & \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^4 \arrow[r, "\iota_*"] & \mathbb{Z}[1/4] \oplus \mathbb{Z}^3 \arrow[d] \\
\mathbb{Z}[1/2]^2 \arrow[u] & 0 \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

Chair

$$\begin{tikzcd}
\mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^4 \arrow[r, "\mathrm{ev}"] & \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^4 \arrow[r, "\iota_*"] & \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z} \arrow[d] \\
\mathbb{Z}[1/2]^2 \arrow[u] & 0 \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

Robinson triangle (Penrose)

$$\begin{tikzcd}
\mathbb{Z}^{37} \arrow[r, "\mathrm{ev}"] & \mathbb{Z}^{40} \arrow[r, "\iota_*"] & \mathbb{Z}^{9} \arrow[d] \\
\mathbb{Z}^{5} \arrow[u] & 0 \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

Chair (variant)[6]

$$\begin{tikzcd}
\begin{array}{c}\mathbb{Z}[1/3]^4 \oplus \mathbb{Z}[1/2]^2 \\ \oplus \mathbb{Z}^{76}\end{array} \arrow[r, "\mathrm{ev}"] & \begin{array}{c}\mathbb{Z}[1/9] \oplus \mathbb{Z}[1/3]^6 \\ \oplus \mathbb{Z}[1/2]^4 \oplus \mathbb{Z}^{101}\end{array} \arrow[r, "\iota_*"] & \begin{array}{c}\mathbb{Z}[1/9] \oplus \mathbb{Z}[1/3]^4 \\ \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{26}\end{array} \arrow[d] \\
\mathbb{Z}[1/3]^2 \arrow[u] & 0 \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

Danzer 7

$$\begin{tikzcd}
\mathbb{Z}^{2017} \arrow[r, "\mathrm{ev}"] & \mathbb{Z}^{2240?} \arrow[r, "\iota_*"] & \mathbb{Z}^{230?} \arrow[d] \\
\mathbb{Z}^{6} \arrow[u] & 0 \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

---

[6] No rotational symmetry, expansion factor of 3. Assuming direct limit splits as direct sums.

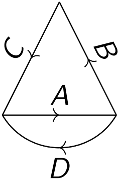
[7] Computed with a script in Sage. Everything is collared.

Thank you!

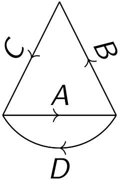
“Thickening”

$$\begin{aligned}A &\xmapsto{\sigma^2} (AD)AB(C)\\ B &\xmapsto{\sigma^2} (B)CAB(C)\\ C &\xmapsto{\sigma^2} (AB)CAD(A)\\ D &\xmapsto{\sigma^2} (B)CAD(A)\end{aligned}$$

C_n^0

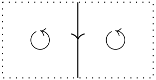


$K_0(AF_n; G^u)$

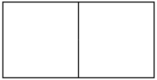


“Thickening”

C_n^1



$K_0(AF_n; G^u)$



Thus we obtain a map $\text{th} : \varinjlim_n (C_n^{d-1}, \sigma^\top) \to K_0(AF; G^u)$.

$d = 1$

For $d = 1$, th is an isomorphism, because there are no relations in relative K_0.

Thus the square

$$\begin{tikzcd}
\varinjlim_n (C_n^0, \sigma^\top) \arrow[r, "\delta^0"] \arrow[d, "\mathrm{th}"] & \varinjlim_n (C_n^1, \sigma^\top) \arrow[d, equal] \\
K_0(AF; G^u) \arrow[r, "\mathrm{ev}"] & K_0(AF)
\end{tikzcd}$$

commutes with vertical maps isomorphisms.

One then adds in kernels and cokernels on both sides.

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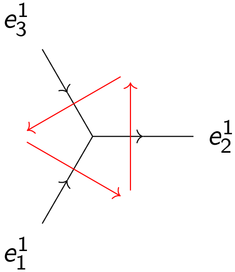
What about $d = 2$?

$d = 2$

Partial homeomorphisms can be composed to yield the identity.



$d = 2$

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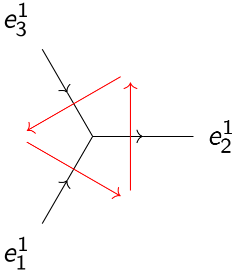

Thus the relations in $K_0(AF_n; G^u)$ are exactly given by $\mathrm{im}\delta_n^0$, and we obtain an isomorphism $\widetilde{\mathrm{th}} : \varinjlim(C_n^1/\mathrm{im}\delta_n^0, \widetilde{\sigma_\top}) \to K_0(AF; G^u)$.

$d = 2$

Fact

$$\begin{tikzcd}
 & & 0 \arrow[r] & \operatorname{im}\iota_* \arrow[dr, hook] & & & \\
0 \arrow[r] & K_1(G^u) \arrow[r, hook] & K_0(AF;G^u) \arrow[r, "\text{ev}"] & K_0(AF) \arrow[r, "\iota_*"] & K_0(G^u) \arrow[r, two heads] & K_1(AF;G^u) \arrow[r] & 0.
\end{tikzcd}$$

$d = 2$

Fact

There exists a splitting

$$\begin{tikzcd}
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\end{tikzcd}$$

$d = 2$

Fact

There exists a splitting

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 & & 0 \arrow[r] & \operatorname{im}\iota_* \arrow[dr, hook] & & & \\
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\end{tikzcd}$$

Anderson-Putnam '98 gives $K_1(AF; G^u) = \check{H}^0(\Omega_T)$.