The Lotka-Volterra Model

- Predator-Prey model from biology
- ODE in 2D
- Analysis of stationary points
- Nullclines to aid drawing the phase portrait
- Lyapunov functions to prove (Lyapunov/asymptotic) stability

$$(*) \quad \begin{array}{ll} \text{prey} & \dot{x} = x(1-y) \\ \text{predator} & \dot{y} = \alpha(x-1)y \end{array} = F\begin{pmatrix} x \\ y \end{pmatrix}, \quad \alpha > 0$$

Only $x, y \geq 0$ are of physical relevance.

More general form $\quad \dot{x} = (A - By)x \qquad A, B, C, D > 0$

$$\dot{y} = (Cx - D)y$$

can be reduced to $(*)$ via a change of coordinates

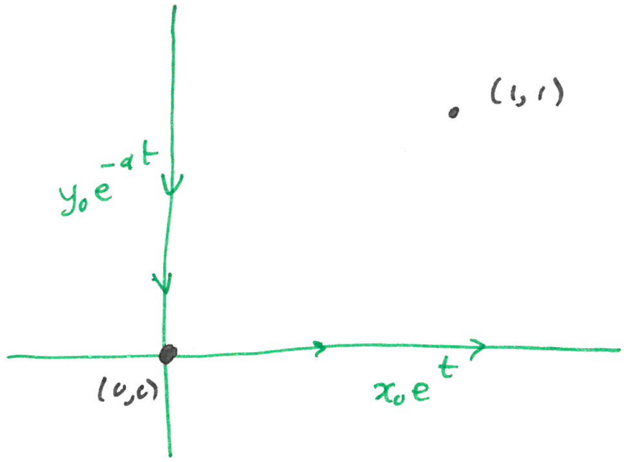
To find stationary points (equilibria)

set $F\begin{pmatrix} x \\ y \end{pmatrix} = 0$

(i) $x = 0 \Rightarrow y = 0$

(ii) $y = 1 \Rightarrow x = 1$

No others.



$$DF_{(x,y)} = \begin{pmatrix} 1-y & -x \\ \alpha y & \alpha(x-1) \end{pmatrix}$$

(i) $DF_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}$ Eigenvalues $-\alpha < 0 < 1$ $\boxed{\text{Saddle.}}$

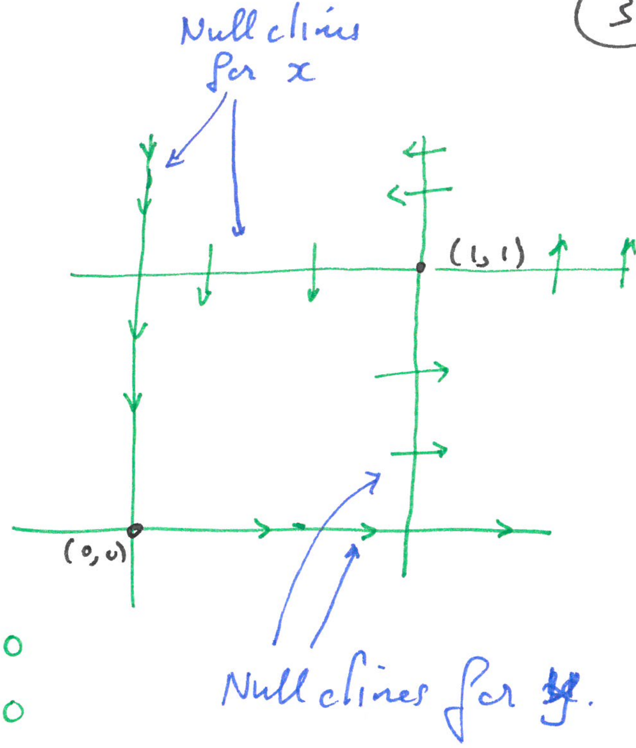
Note $x = 0 \Rightarrow \begin{cases} \dot{x} = 0 \Rightarrow x(t) \equiv 0 \quad \forall t \\ \dot{y} = -\alpha y \Rightarrow y(t) = y_0 e^{-\alpha t} \end{cases}$

$y = 0 \Rightarrow \begin{cases} \dot{y} = 0 \Rightarrow y(t) \equiv 0 \quad \forall t \\ \dot{x} = x \Rightarrow x(t) = x_0 e^{t} \end{cases}$

(ii) $DF_{(1,1)} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}$ Eigenvalues $\lambda = \pm i\sqrt{\alpha}$. $\boxed{\text{Center}}$

If the ODE was linear, then periodic solutions around $(1,1)$, but in the nonlinear case we don't know yet.

Def A <u>nullcline</u> is a region where one component of a vector field F is $=$ zero.



Here: $x=0 \vee y=1$ for $F_1=0$
$y=0 \vee x=1$ for $F_2=0$

This suggest that solutions turn counterclockwise in the 1^{st} quadrant, but we don't know yet if they spiral inwards to $(1,1)$, or outwards away from $(1,1)$, or are periodic.

From
$$\dot{x} = x(1-y)$$
$$\dot{y} = \alpha(x-1)y \qquad (*)$$

try to eliminate time t, and set $y = y(x)$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \alpha \, \frac{x-1}{x} \cdot \frac{y}{1-y}$$

Separate variables

$$\frac{1-y}{y} \dot{y} = \alpha \frac{x-1}{x} \dot{x}$$

Integrate

$$\log y - y + \text{Const} = \alpha(x - \log x)$$

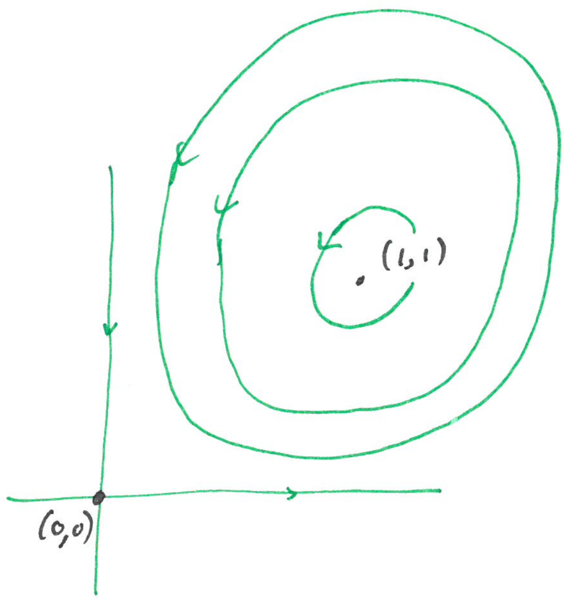
Introduce $f(u) = u - 1 - \log u$

$f(1) = 0$
$f'(s) = 1 - \frac{1}{s} = 0$ at $s=1$
1 is global minimum of f.

$$L(x,y) = \alpha f(x) + f(y) = \text{Const.}$$

So L is a preserved quantity

Solutions of $(*)$ need to remain on level curves of L



Most solutions are therefore periodic.

Compute

$$\frac{d}{dt} L(x,y) = \begin{pmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \end{pmatrix} \bullet \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \overbrace{\nabla L \bullet F}^{\text{Lie derivative}}$$

gradient of L dot-product   vector field

$$= \begin{pmatrix} \alpha\left(1 - \frac{1}{x}\right) \\ 1 - \frac{1}{y} \end{pmatrix} \bullet \begin{pmatrix} x(1-y) \\ \alpha(x-1)y \end{pmatrix}$$

$$= \alpha(x-1)(1-y) + (y-1)\cdot\alpha(x-1) = 0.$$

Def A function $L : \mathcal{U} \to \mathbb{R}$ on a nbh of a stationary point p is called <u>Lyapunov function</u> if

1) $L(p) = 0$, $L(x) > 0$ for $x \in \mathcal{U}\setminus\{p\}$

2) $L(\varphi^{t_1}(x)) \leq L(\varphi^{t_0}(x))$ $\forall x \in \mathcal{U}$ $\forall t_1 > t_0$

(if "$<$" for $\forall x \in \mathcal{U}\setminus\{p\}$ $\forall t_1 > t_0$, then it is a <u>strict</u> Lyapunov function)

Remark

If $L(p) = 0$ and the Lie derivative of L is zero on a nbh $\mathcal{U}$ of p, then L is a Lyapunov function.

If the Lie derivative is < 0 on $\mathcal{U} \setminus \{p\}$ the L is a strict Lyapunov function

Theorem a) If an equilibrium p has a Lyapunov function, then p is Lyapunov stable

b) If the Lyapunov function is strict, then p is asymptotically stable.

Remark i) We give the proof for when Lyapunov function L has non-positive / negative Lie derivative.

ii) Recall

Lyapunov stable : $\forall \varepsilon > 0 \ \exists \delta > 0 \quad z(0) \in B_\delta(p) \Rightarrow$
$z(t) \in B_\varepsilon(p) \quad \forall t \geq 0$

Asymptotically stable $\exists \delta > 0 \quad z(0) \in B_\delta(p) \Rightarrow$
$z(t) \to p$ as $t \to \infty$.

Proof a) Since $\frac{d}{dt} L(z(t)) \leq 0$, we have $L(z(t_1)) \leq L(z(t_0))$ $\forall t_1 \geq t_0$.

L has a local minimum at p, so $\forall \varepsilon > 0$ $\exists \eta > 0$ such that $\{z : L(z) < \eta\} \subset B_\varepsilon(p)$, and $\exists \delta > 0$ such that $B_\delta(p) \subset \{z : L(z) < \eta\}$.

Therefore $z(t) \in B_\varepsilon(p)$ for all $z(0) \in B_\delta(p)$.

This is Lyapunov stability.

b) Let $z(0) \in B_\delta(p)$ as before.

Take $\eta > 0$ so small that

$$V = \overline{\{w \in B_\delta(p) : L(w) \leq \eta\}} \subset B_\delta(p)$$

Then V is invariant under the flow and compact.

Since the Lie derivative < 0 on $B_\delta(p) \setminus \{p\}$,

$$\overline{\{w(t) : t \geq 1\}} \not\ni w(0) \qquad \forall w(0) \in B_\delta(p) \setminus \{p\}$$

(because L is continuous). In fact $\exists W \ni w(0)$ such that $\boxed{\varphi^t(W) \cap W = \emptyset \quad \forall t \geq 1}$ (*)

By compactness of V, $\exists w \in V$ and $t_n \to \infty$ such that $z(t_n) \to w$. But then $w = p$, otherwise (*) fails ∎