Complexity of η-od-like continua

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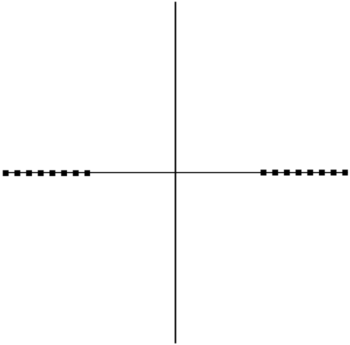
A continuum is a compact connected metric space. W. Lewis asked in *Indecomposable Continua. Open problems in topology II,* whether there exists, for every $\eta \geq 2$, an atriodic simple $(\eta + 1)$-od-like continuum which is not simple η-od-like and, if such continuum exists, whether it has a variety of properties such as being planar or being an arc-continuum, among others. Some partial results have been obtained by W.T. Ingram, P. Minc, C.T. Kennaugh and L. Hoehn.

In the following sections we will develop the notion of a combinatorial η-od cover of a graph, a tool which may enable one to prove that certain examples of continua are not η-od-like.

We will suggest the construction of an atriodic simple $(\eta+1)$-od-like continuum which is not simple η-od-like and has properties such as being planar, being an arc-continuum, and span zero.

Let $\eta \in \mathbf{N}$ be such that $\eta \geq 3$.



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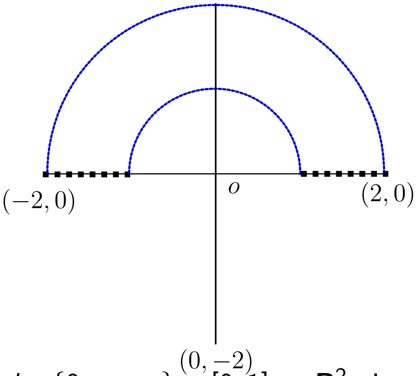

Define the function $b : \{0, \ldots, \eta\} \times [0, 1] \to \mathbf{R}^2$ given by

$$b(i, t) = \begin{cases} (0, -1 - t) & \text{if } i = 0, \\ ((1 + t) \cos \frac{(i-1)\pi}{\eta - 1}, (1 + t) \sin \frac{(i-1)\pi}{\eta-1}) & \text{if } i \neq 0. \end{cases}$$

For each $i \in \{0, \ldots, \eta\}$, define $B_i = \{b(i, t) : t \in [0, 1]\}$.

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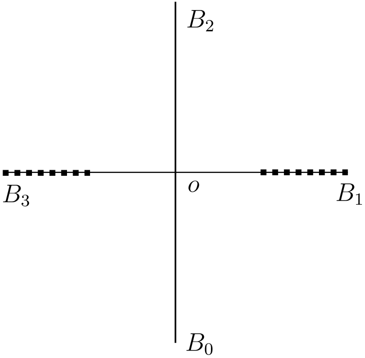

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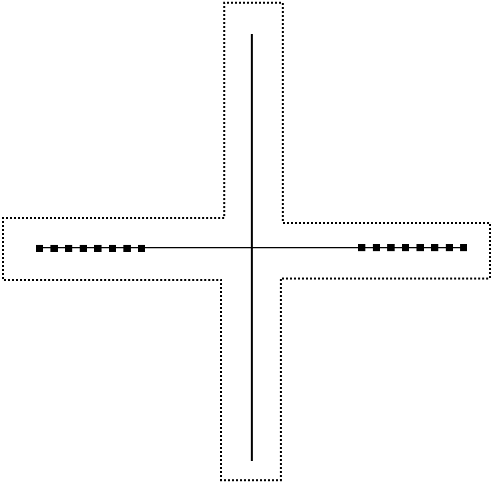
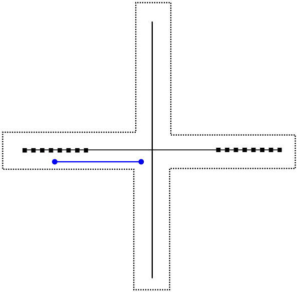
For each $i \in \{0, \ldots, \eta\}$, define $B_i = \{b(i,t) : t \in [0,1]\}$.
In the set $\Gamma = B_0 \cup \cdots \cup B_\eta \cup \{o\}$ we define the relation $p \cong q$ if and only if $p = q$ or $\{p, q\} \subset B_i$ for some $i \in \{0, \ldots, \eta\}$.



Definition 1

A function $\omega : V(G) \to \Gamma$ is called compliant if for every vertices u and v of G we have that $\omega(u) \cong o$ and $\omega(v) \cong b(i,1)$, or $\omega(u) \cong b(i,1)$ and $\omega(v) \cong o$; for some $i \in \{0, \ldots, \eta\}$.



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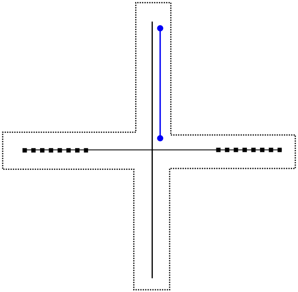

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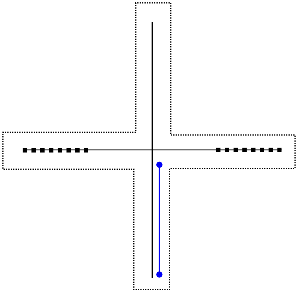

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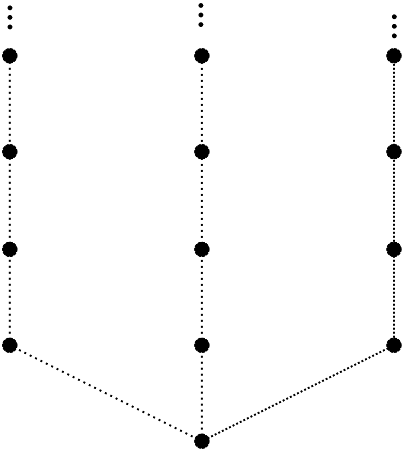
Definition 2

Let $\epsilon > 0$ and ω a compliant function. A (T_0, ϵ)-projection via ω is a continuous function $\Omega : G \to T_0$ such that:
(1) Ω extends ω,
(2) if u and v are adjacent vertices of G, $\Omega|uv$ is a homeomorphism between uv and $\omega(u)\omega(v)$,
(3) for every $p \in G$, $d_2(p, \Omega(p)) < \epsilon$.

We will work with the infinite η-od $\mathcal{C}$ defined by



Construction of $\mathcal{C}$ with $\eta = 3$.

Definition 3

Let $\delta > 0$. A δ-combinatorial η-od cover for a compliant function ω is a function $f : V(G) \to V(\mathcal{C})$ such that for any vertices u, v, v_1, v_2, v_3 of G we have the following properties

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CIII. Assume that v_1, v_2, v_3 are consecutive vertices in G, $v \notin \{v_1, v_2, v_3\}$, $f(v_1) \neq f(v_3)$, $f(v) = f(v_2)$, $f(v_2)$ is in the path of $\mathcal{C}$ between $f(v_1)$ and $f(v_3)$, and $f(v_2)$ is not the branch vertex of $\mathcal{C}$. Assume further that $0 \leq s < t \leq 1$, and for some $i \in \{0, \ldots, \eta\}$, $\omega(v_2) = b(i, s)$ and $\omega(v) = b(i, t)$. Then $t - s < \delta$.

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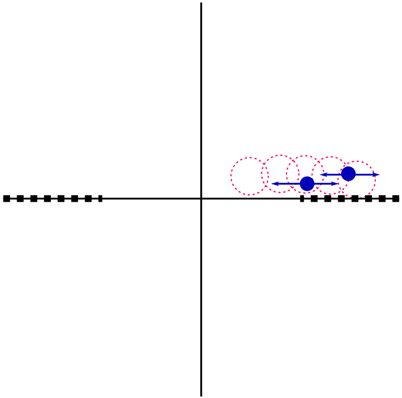
Proposition 4

Let small δ, $\epsilon \in (0, \frac{\delta}{2})$, and ω a compliant function with a (T_0, ϵ)-projection Ω. If G has an open η-odic cover of mesh less than $\delta - 2\epsilon$, then ω has a δ-combinatorial η-od cover.

Property CI.

If $f(u) = f(v)$, then $\omega(u) \cong \omega(v)$.



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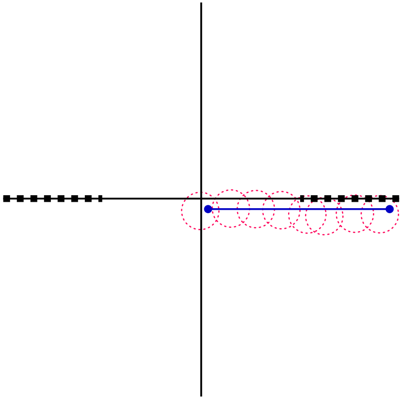

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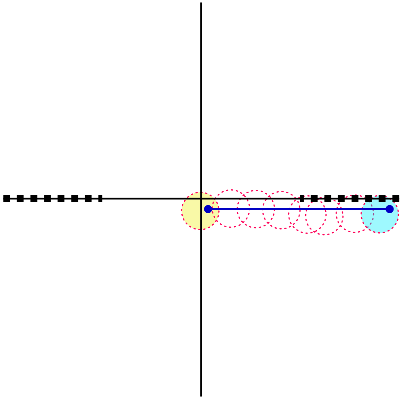

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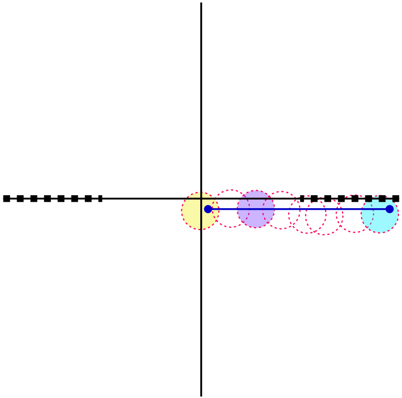

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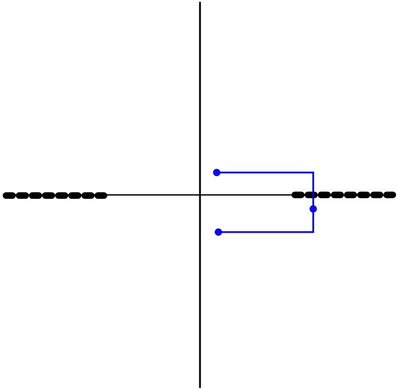

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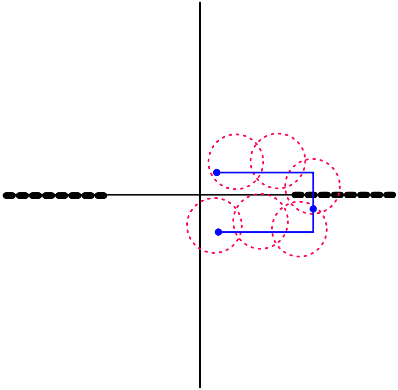

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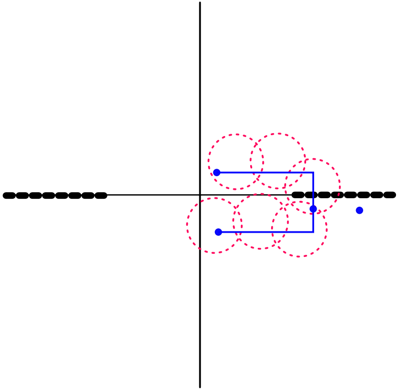

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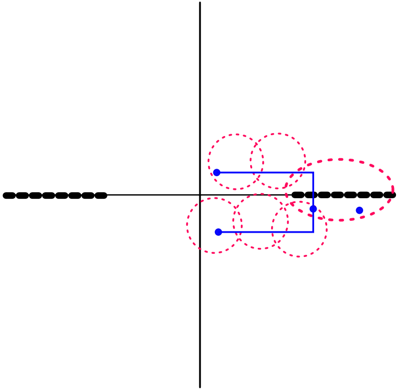

Let small δ and $\epsilon \in (0, \frac{\delta}{2})$. Also, let X be a continuum defined as the limit of a sequence of graphs $\langle T_n \rangle_{n=1}^{\infty}$, each described by a (T_0, ϵ)-projection via ω_N (where ω_N is a compliant function for the graph T_n). With the Proposition 4, we will be able to conclude that X cannot be covered by an open η-odic cover with mesh less than $\delta - 2\epsilon$, if for every graph T_n we have that there doesn't exists a δ-combinatorial η-od cover for ω_N.

Definition 5 (Lelek,[5])

Let X be a continuum with metric d_X, the span of X, denoted by σX, is the supreme of every $0 \leq \gamma$ for which there exists a continuum $Z \subseteq X \times X$ such that:

1. $\gamma \leq d_X(x, y)$ *for every* $(x, y) \in Z$.
2. $\pi_1(Z) = \pi_2(Z)$ *where* $\pi_1, \pi_2 : X \times X \to X$ *are the projections of the first and second coordinate, respectively.*

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Lema 6

Let $\delta > 0$. If X is a simple δ-$(\eta + 1)$-od with metric d_X, then $\sigma X \leq \delta$.

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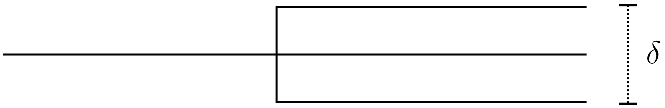

From [7] we have the following result,

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These known result, together with Lemma 6, will allow us to conclude that a continuum X built as the nested intersection of small neighborhoods of simple δ_N-$(\eta + 1)$-ods, such that $\lim_{n\to\infty} 0$, will have $\sigma X = 0$ and, therefore, will be atriodic.

The construction of a continuum X which is $(\eta + 1)$-od-like, for which we will prove the following properties:
(1) is atriodic, and
(2) is not η-od-like,
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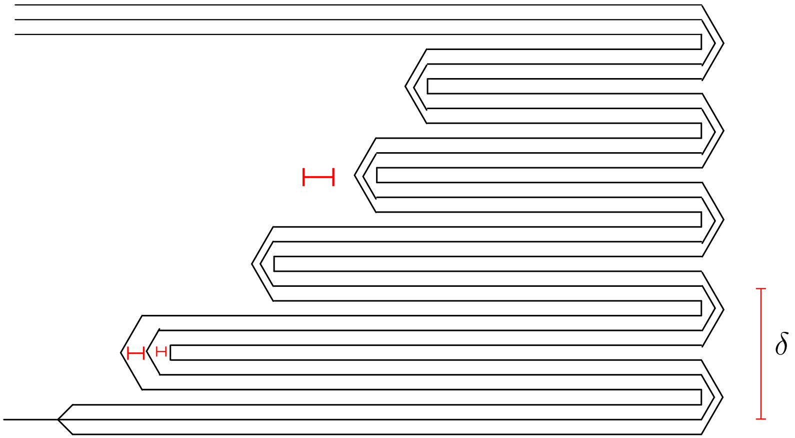
We have designed a sequence of $(\eta+1)$-ods T_n for which we will prove the following properties:
(I) $\lim_{n\to\infty} \sigma T_n = 0$, and
(II) it cannot be covered by an open η-odic cover with small mesh.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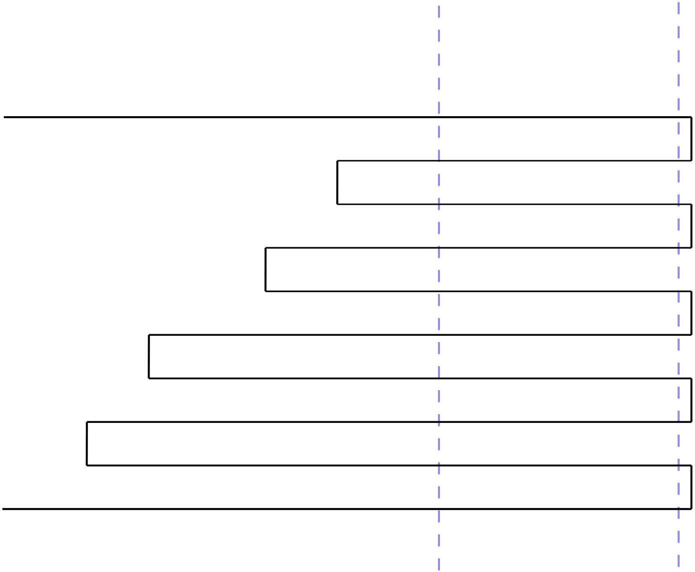

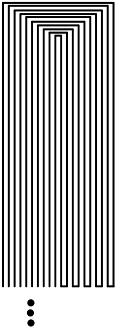

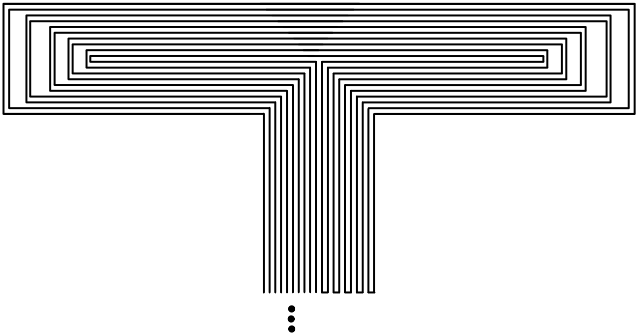

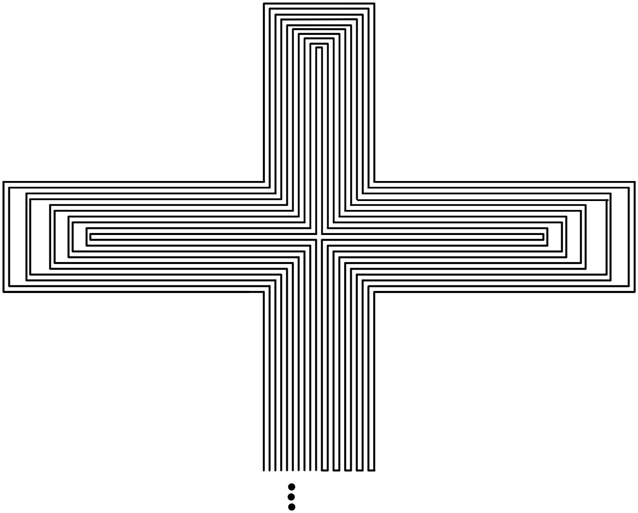

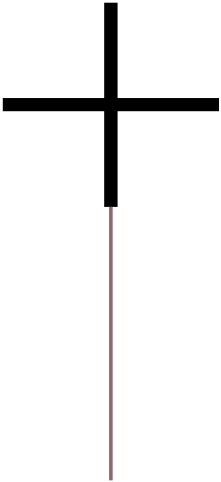

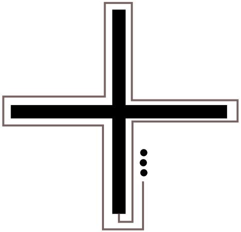

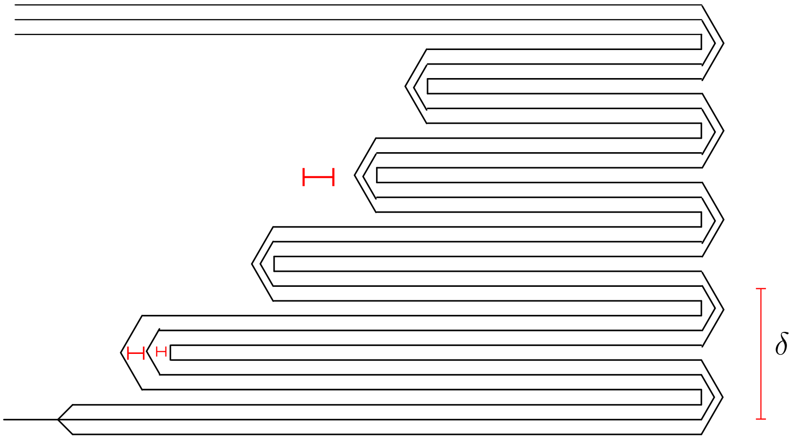
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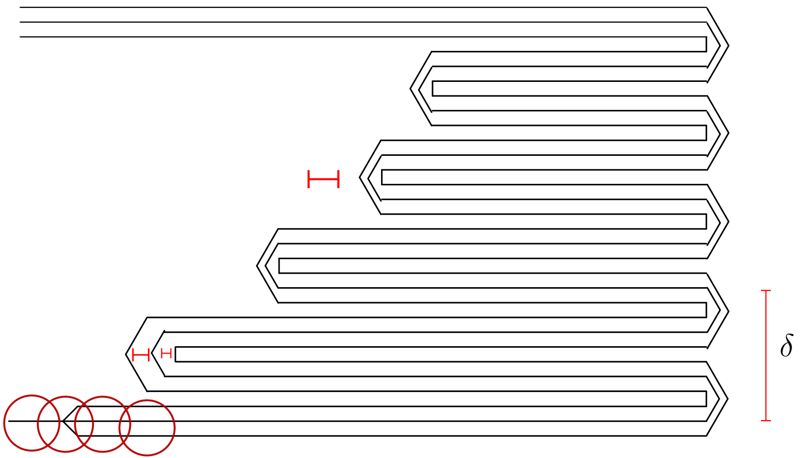
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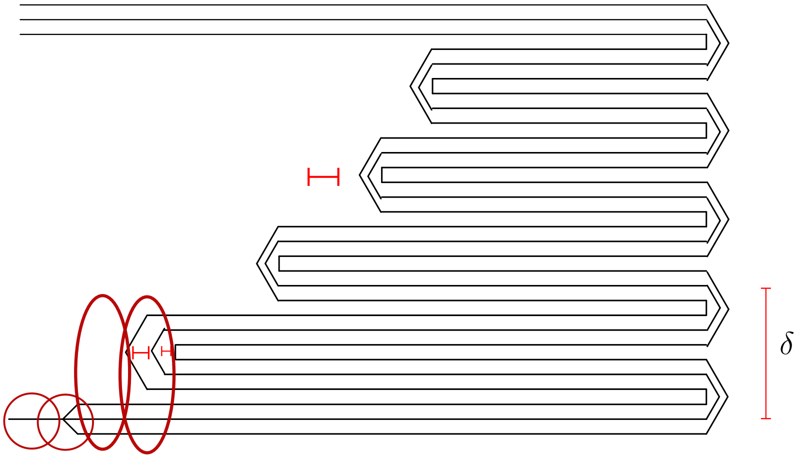
δ

L. C. Hoehn, *A non-chainable plane continuum with span zero*, Fund. Math. 211 (2011), no. 2, 149–174. MR 2747040

W. T. Ingram, *An atriodic tree-like continuum with positive span*, Fund. Math. 77 (1972), no. 2, 99–107. MR 0365516

——-, *Hereditarily indecomposable tree-like continua*, Fund. Math. 103 (1979), no. 1, 61–64. MR 535836

C. T. Kennaugh, *Complexity of atriodic continua*, Ph.D. thesis, Texas Tech University, 2009.

A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. 55 (1964), 199–214

Wayne Lewis, *Indecomposable continua, Open problems in topology. II* (Elliott Pearl, ed.), Elsevier B. V., Amsterdam, 2007, pp. 303–317. MR 2367385

Piotr Minc, *An atriodic simple-4-od-like continuum which is not simple-triod-like,* Trans. Amer. Math. Soc. 338 (1993), no. 2, 537–552. MR 1197564 2

THANK YOU