On tilings, amenable equivalence relations and foliated spaces.

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Joint work with:

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- Two motivating questions
- The space of tilings
- A tiling on $Sol(a,b)$
- Its continuous hull and its foliated structure.
- An amenable equivalence relation which is not affable.

Two motivating questions

G. Hector : M compact metrizable space.

$\mathcal{F}$ foliation on M.

Are the leaves of $\mathcal{F}$ quasi-isometric to Cayley graphs?

True when generic leaves have two ends.

(Blanc / Ghys)

Giordano - Putnam - Skau

minimal and free continuous action of an amenable countable group on the Cantor set

$\overset{\text{O.E}}{\sim}$

minimal $\mathbb{Z}$-action on the Cantor set

Giordano - Putnam - Skau

minimal and free continuous action of an amenable countable group on the Cantor set $\overset{O.E}{\sim}$ minimal $\mathbb{Z}$-action on the Cantor set

Broader problem: What properties of a group can you 'read' from the equivalence relations induced by its actions?

The Giordano-Putnam-Skau conjecture is the analog, in the topological setting, of the Connes-Feldman-Weiss theorem for measurable actions.

What we did:

Construct a family of examples of

* foliations of compact spaces, where leaves are not quasi-isometric to Cayley graphs.

* equivalence relations on the Cantor set which are amenable but do not come from a $\mathbb{Z}$-action.

The space of tilings

Tilings on a Lie Group G.

- Finite set of tiles.
- No overlaps or gaps.
- Periodic tiling : repeating pattern.
- Aperiodic tiling : no translation leaves it invariant.
- Repetitive tiling : for any R there is an R' such that any ball of radius R has a translation copy in any ball of radius R'.
- A condition on how faces touch.

Space of tilings:

$(\mathcal{T}, o)$

$\downarrow$ tiling

$\hookrightarrow$ marked point

Topology:

$\mathcal{T}$ and $\mathcal{T}'$ are close if big balls centered at O are close (they differ by an isometry close to id)

Space of tilings:

$(\mathcal{T}, 0)$

tiling

marked point

G acts by translations.

Topology:

$\mathcal{T}$ and $\mathcal{T}'$ are close if big balls centered at 0 are close (they differ by an isometry close to id)

Continous hull of $(\mathcal{T}, 0)$: closure of its G-orbit.

When $\mathcal{T}$ is repetitive and aperiodic, its continuous hull has a foliated structure.

locally $U \times$ Cantor set

$U \subset G$ open set

leaves $\sim G$

all leaves are dense

A tiling on Sol(a,b)

$a, b > 0$

$Sol(a,b) = \mathbb{R}^2 \rtimes \mathbb{R}$ defined by the $\mathbb{R}$-action

$$z \in \mathbb{R} \longmapsto \begin{pmatrix} e^{az} & 0 \\ 0 & e^{-bz} \end{pmatrix} \in GL_2(\mathbb{R})$$

That is,

$$Sol(a,b) = \left\{ \begin{pmatrix} e^{az} & x & 0 \\ 0 & 1 & 0 \\ 0 & y & e^{-bz} \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

When $a = b$, $\mathrm{Sol}(a,b) = \mathrm{Sol}$.

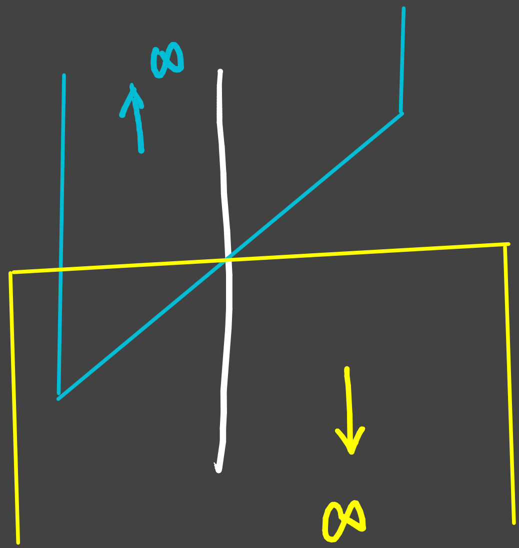
$$\begin{pmatrix} e^{z/2} & x & 0 \\ 0 & 1 & 0 \\ 0 & y & e^{-z/2} \end{pmatrix}$$

$$e^{-2z}\,dx^2 + e^{2z}\,dy^2 + dz^2$$

is left-invariant.

hyperbolic planes

hyperbolic planes



When $a \neq b$

$$\begin{pmatrix} e^{az/2} & x & 0 \\ 0 & 1 & 0 \\ 0 & y & e^{-bz/2} \end{pmatrix}$$

hyperbolic planes

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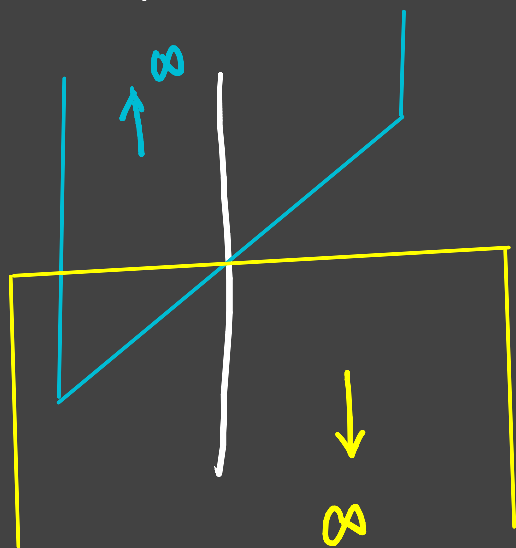


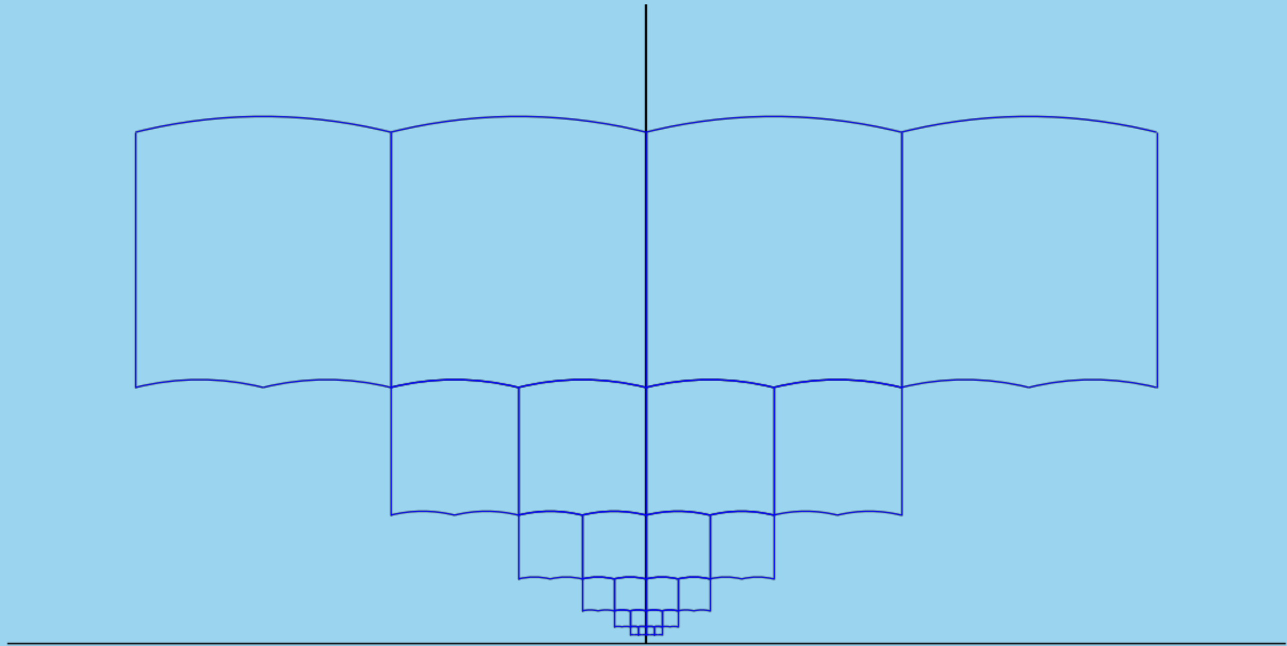
Samuel Petite: On invariant measures of finite affine type tilings.
(2006)

Tilings on the affine group $\mathrm{Aff}_+(\mathbb{R})$.

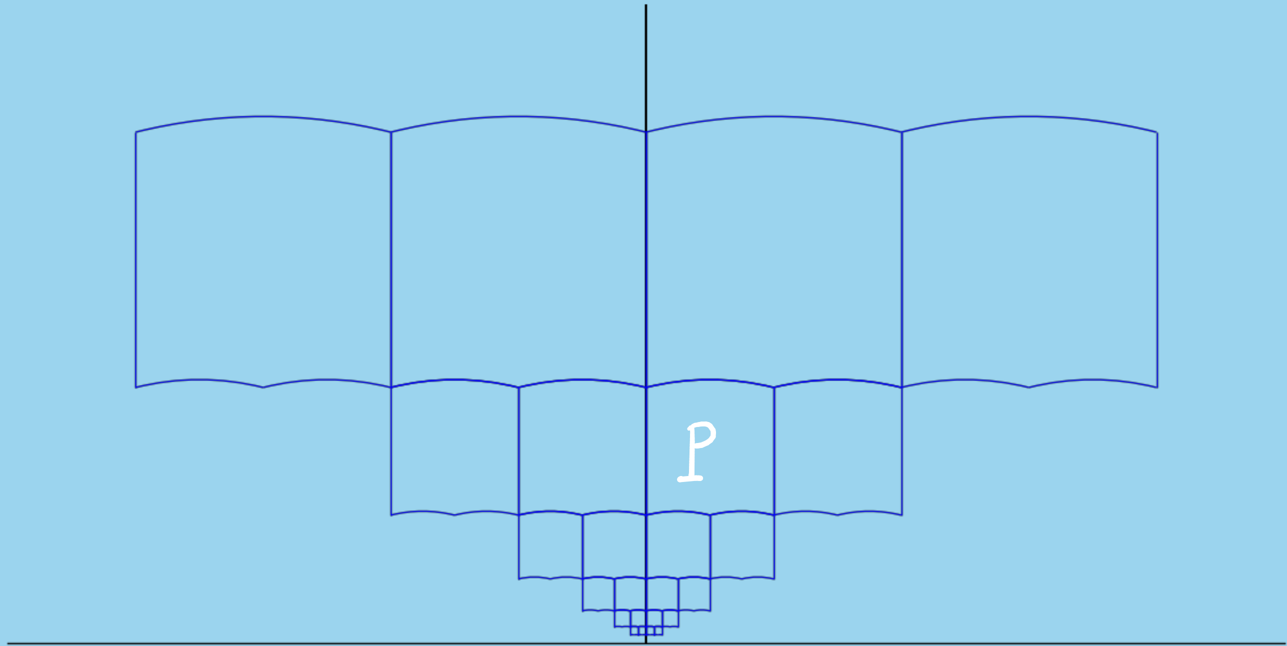
Samuel Petite: On invariant measures of finite affine type tilings.
(2006)

<u>Thm.</u> (Eskin, Fisher, Whyte, 2012)

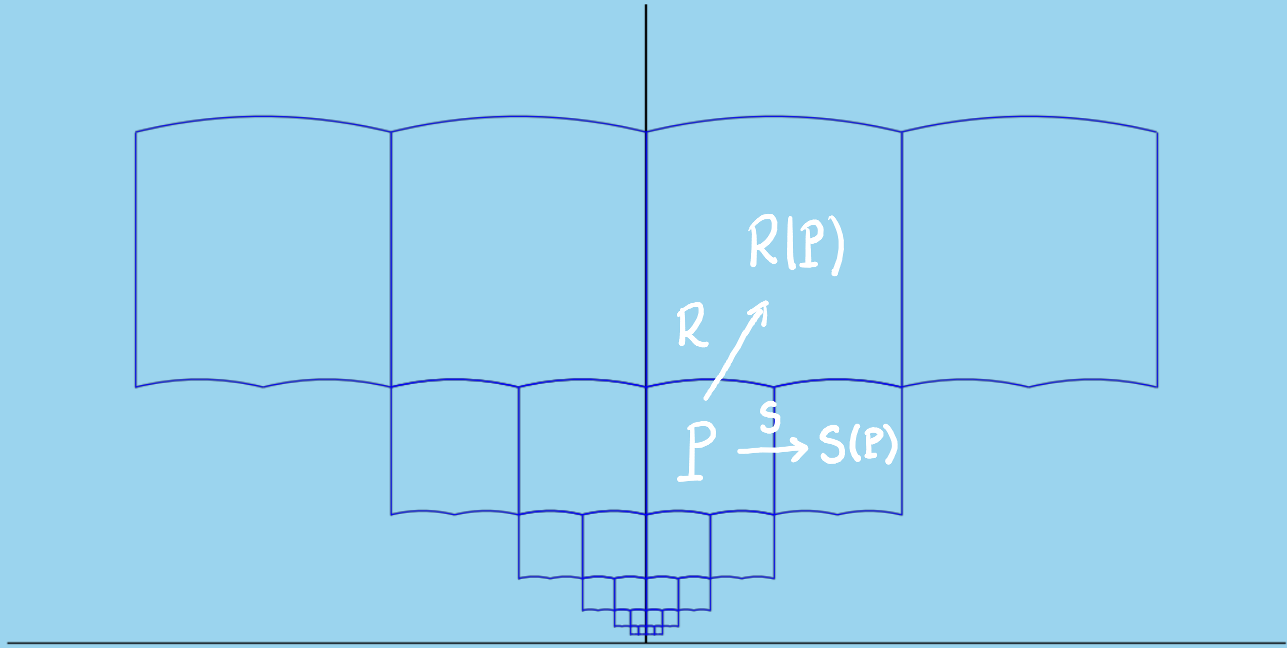
$a \neq b \implies Sol(a,b)$ is not quasi-isometric to a Cayley graph.



We start by a tiling of the hyperbolic plane.

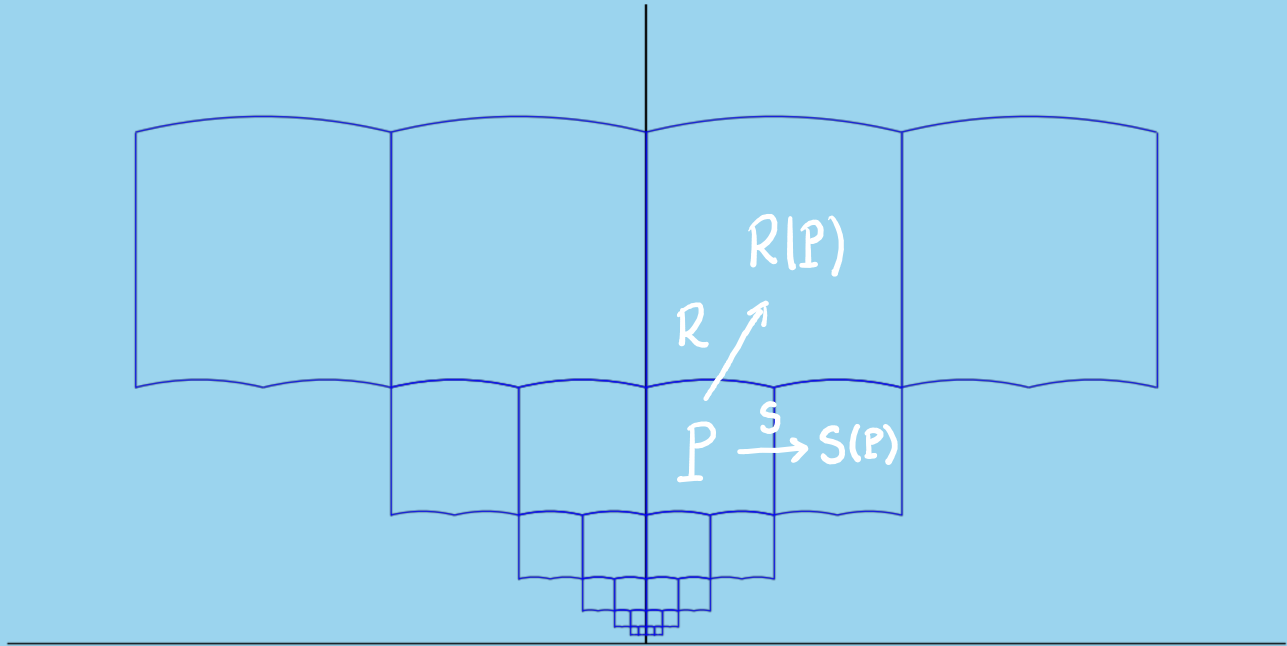


Only one tile P.



$R: z \longmapsto 2z \quad , \quad S: z \longmapsto z+1$

$\mathcal{T} = \{ R^i \circ S^j (P) : i,j \in \mathbb{Z} \}$



It is neither periodic nor aperiodic.

Symmetry : $\langle R \rangle$

We color the tiling to break its symmetry, in a way which makes it aperiodic and recurrent.

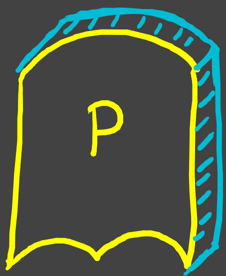
<u>Rmk</u>: $Aff_+(\mathbb{R}) \simeq$ hyperbolic plane.

T can be seen as a repetitive and aperiodic tiling of $Aff_+(\mathbb{R})$.

$Sol(a,b)$ is $\mathbb{R}^3$ with the metric

$$\underline{e^{-2az}\,dx^2} + \underline{e^{2bz}\,dy^2} + \underline{\underline{dz^2}}.$$

Two transverse families of hyperbolic planes.



We 'fatten' P along the direction of horocycles in the blue direction.

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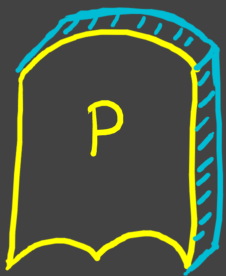


$$\hat{T}_s(x,y,z) = (0,s,0)\cdot(x,y,z) =$$
$$= (x, y+s, z)$$

$$\boxed{\hat{P} = \bigcup_{s\in[0,1]} \hat{T}_s(P).}$$

Construction of a tiling of Sol(a,b)

$\hat{P}$ initial tile

Translate $\hat{P}$ along 3 directions

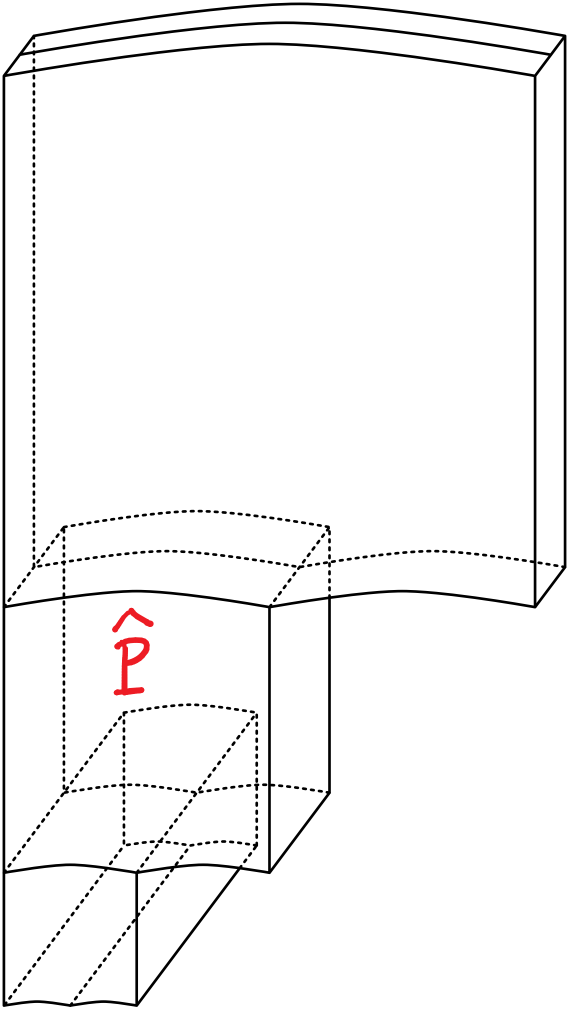
- $\hat{R}(x,y,z) = (0,0,\log(2)/a)\cdot(x,y,z) = (2x, \Delta y, z+\log(2)/a)$
- $\hat{S}$ is similar
- $\hat{T}_1$

$$\Delta = 2^{-b/a}$$

We obtain

$$\widehat{\mathcal{T}} = \left\{ \hat{R}^i \circ \hat{S}^j \circ \hat{T}_k (\hat{P}) \ , \ i,j,k \in \mathbb{Z} \right\}$$

<u>Thm.</u> $\widehat{\mathcal{T}}$ is a tiling of $\mathrm{Sol}(a,b)$.
If $b/a = \log(n)/\log(2)$ for some $n \in \mathbb{Z}$, $\widehat{\mathcal{T}}$ is face-to-face and repetitive. With an appropriate colouring, it also becomes aperiodic.


$\hat{P}$

The continuous hull of $\hat{\mathcal{T}}$.

$b/a = \log(n) / \log(2)$

When $\mathrm{Sol}(a,b)$ acts by translation on the space of tilings

$$\mathcal{M}_{a,b} = \overline{\mathrm{Sol}(a,b)\cdot\mathcal{T}}$$

It is compact.

Thm. If $\pm b/a = \log(n)/\log(2)$ for some $n \in \mathbb{Z}$, $\mathcal{M}_{a,b}$ is a nonempty compact metrizable space endowed with a free and minimal action of $\mathrm{Sol}(a,b)$. With the foliation given by the orbits, it is a lamination which is transversally Cantor.

Thm. If $\pm b/a = \log(n)/\log(2)$ for some $n \in \mathbb{Z}$, $\mathcal{M}_{a,b}$ is a nonempty compact metrizable space endowed with a free and minimal action of $Sol(a,b)$. With the foliation given by the orbits, it is a lamination which is transversally Cantor.

Rmk: The leaves of $\mathcal{M}_{a,b}$ are not quasi-isometric to Cayley graphs.

An amenable equivalence relation which is not affable

It doesn't come from a continuous $\mathbb{Z}$-action.

GPS conjecture:

G countable and amenable acting freely and minimally by homeos on the Cantor set $\Rightarrow$ (X, G) topologically O.E. to a $\mathbb{Z}$-action.

The transversal of $\mathcal{M}_{a,b}$ is the Cantor set.

$x \mathcal{R} y \iff x, y$ belong to the same leaf.

$$\boxed{\text{Then } \mathcal{R} \text{ is amenable but doesn't come from a continuous } \mathbb{Z}\text{-action.}}$$

<u>Why?</u>

$Sol(a,b)$ solvable $\Longrightarrow$ $Sol(a,b)$ amenable $\Longrightarrow$

$\Longrightarrow$ $\mathcal{R}$ amenable.

$Sol(a,b)$ not unimodular $\Longrightarrow$ $\mathcal{M}_{a,b}$ does not admit a transverse invariant measure

Thank you for listening!