The spaces of non-descendible quasimorphisms and bounded characteristic classes

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Def Γ : group

- $\mu : \Gamma \to \mathbb{R}$: <u>homogeneous quasimorphism</u>

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \cdot\ \exists D \geq 0 \text{ s.t. } \forall g, h \in \Gamma,\ |\mu(gh) - \mu(g) - \mu(h)| \leq D \\ \cdot\ \forall g \in \Gamma,\ \forall n \in \mathbb{Z},\ \mu(g^n) = n \cdot \mu(g) \end{cases}$$

- $Q(\Gamma) = \{ \mu : \Gamma \to \mathbb{R} : \text{homogeneous quasimorphism} \}$

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{Homeo}}_+(S^1) \to \mathrm{Homeo}_+(S^1) \to 1$$

$Q(\widetilde{\mathrm{Homeo}}_+(S^1)) \cong \mathbb{R}$ (spanned by Poincaré's translation number)

$H^2(\mathrm{Homeo}_+(S^1); \mathbb{R}) \cong \mathbb{R}$ (spanned by the Euler class)

group cohomology

Hence $Q(\widetilde{\mathrm{Homeo}}_+(S^1)) \cong H^2(\mathrm{Homeo}_+(S^1); \mathbb{R})$.

In general, $Q(\widetilde{G}) \not\cong H^2(G)$

for a topological group G and its universal covering $\widetilde{G}$.

(e.g. $Q(\widetilde{\mathrm{Diff}}_+(S^1)) \cong \mathbb{R}$ but $H^2(\mathrm{Diff}_+(S^1)) \not\cong \mathbb{R}$)

Thm (Kawasaki - M.)

G : connected topological group

$p : \widetilde{G} \to G$: universal covering

Then, $Q(\widetilde{G}) \big/ H^1(\widetilde{G}) + p^*Q(G)$ is isomorphic to

$$\mathrm{Im}\left(B\iota^* : H^2_s(BG) \to H^2(G)\right) \cap \mathrm{Im}\left(c_G : H^2_b(G) \to H^2(G)\right)$$

singular cohomology bounded cohomology

$Q(\tilde{G}) \big/ H^1(\tilde{G}) + p^*Q(G)$

$Q(\tilde{G}) \supset H^1(\tilde{G}) := \{ h : \tilde{G} \to \mathbb{R} : \text{homomorphism} \}$: trivial h.qm

$Q(\tilde{G}) \supset p^*Q(G)$: h.qms on $\tilde{G}$ descendible to G

$$\left(\begin{array}{l} \mu \in Q(\tilde{G}) \\ \quad \text{is in } p^*Q(G) \end{array} \iff \begin{tikzcd} \tilde{G} \arrow[r, "\mu"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \\ G \arrow[ur, dashed, "\exists\, h.qm"'] & \end{tikzcd} \right)$$

$$\boxed{Q(\tilde{G}) \big/ H^1(\tilde{G}) + p^*Q(G) : \text{the space of non-descendible homogeneous quasimorphisms}}$$

<u>Rmk</u>

$Q(\tilde{G}) \ni \mu : \tilde{G} \to \mathbb{R}$: NOT necessarily continuous.

$\mathrm{Im}\left(B\iota^* : H_s^2(BG) \to H^2(G)\right) \cap \mathrm{Im}\left(c_G : H_b^2(G) \to H^2(G)\right)$

G : topological group

G^δ : group G with discrete topology

- $\iota : G^\delta \to G$: identity homomorphism

  $B\iota^* : H_s^2(BG) \longrightarrow H_s^2(BG^\delta) \cong H^2(G)$

  $H^2(G) \supset \mathrm{Im}\, B\iota^*$: the space of characteristic classes of foliated G-bundles

- $H_b^*(G)$: bounded cohomology of G

  $c_G : H_b^2(G) \longrightarrow H^2(G)$: comparison map

  $H^2(G) \supset \mathrm{Im}\, c_G$

$\mathrm{Im}\, B\iota^* \cap \mathrm{Im}\, c_G$: the space of bounded characteristic classes of foliated G-bundles

Applications

$$\begin{array}{ccc} & S^2 \times S^2 & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ S^2 & & S^2 \end{array}$$

$\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)$: symplectomorphism group

$\mathrm{Cont}_0(S^3, \xi)$: contactomorphism group

standard contact structure

$\omega_\lambda = p_1^*\omega + \lambda \cdot p_2^*\omega$

area form on S^2

$1 < \lambda \leq 2$

$\alpha \in H_s^2(B\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)) \cong \mathbb{R}$

$\beta \in H_s^2(B\mathrm{Cont}_0(S^3, \xi)) \cong \mathbb{R}$

: "primary obstruction class"

Cor

$B\tau^*\alpha \in H^2(\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda))$: non-zero and bounded

$B\tau^*\beta \in H^2(\mathrm{Cont}_0(S^3, \xi))$: non-zero and unbounded

Key

- $\exists \mu \in Q(\widetilde{\mathrm{Symp}}_0(S^2 \times S^2, \omega_\lambda))$: non-descendible h.qm (Ostrover '06)
- $Q(\widetilde{\mathrm{Cont}}_0(S^3, \xi)) = 0$ (Fraser-Polterovich-Rosen '18)