Topological group actions by group automorphisms and Banach representations

Michael Megrelishvili

Bar-Ilan University

This project is dedicated to Vladimir Pestov
on the occasion of his 65th birthday

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3. E. Glasner and M. Megrelishvili, *Representations of dynamical systems on Banach spaces.* Survey paper in: Recent Progress in General Topology III, Springer, Atlantis Press, 2014.
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Main Question of this talk

Q1: What is the dynamical complexity of the **conjugation action**

$$G \curvearrowright G_c, \quad (g,x) \mapsto gxg^{-1}$$

for locally compact (second countable) topological groups G?

Q2: When this action is a "part" of the natural dual action $\mathrm{Iso}\,(V) \curvearrowright (B_{V^*}, w^*)$ via some continuous representation $h\colon G \to \mathrm{Iso}\,(V)$ for low complexity Banach spaces V?

Remark: Easy for left regular actions $G \curvearrowright G$ with G $\in$ LC.

(Gelfand–Raikov) $\forall G \in LC \quad G \in \mathbf{Hilb^r}$ (Hilbert representable). $\downarrow$

One may derive that for every separable metrizable locally compact G the **left regular action** $G \times G \to G, (g,x) \mapsto gx$ is $\mathbf{Hilb^r}$.
Sketch: if $h\colon G \hookrightarrow \mathrm{Is}\,(H)$, then $\exists$ countably many G-maps $f_n\colon G \to \frac{1}{2^n}B_H, g \mapsto gv_n$ s.t. the diagonal G-map $G \hookrightarrow \oplus_{n\in\mathbb{N}} H_n$ is a uniform G-embedding. This induces a proper representation of $G \curvearrowright G$ on the l^2-sum of countably many copies of H.

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Conjugation action case

In contrast,
for the action by conjugations $G \curvearrowright G_c, (g,x) \mapsto gxg^{-1}$ the representation theory and the corresponding hierarchy is widely open even for classical (matrix) LC groups.

Lemma: For every topological group G and a continuous action $G \times H \to H$ on a topological group H by automorphisms the G-space H has a proper G-compactification $H \hookrightarrow K$.

$\downarrow$ (use Teleman's thm)
Proposition: Every conjugation action has a proper Banach representation on $C(K)$.

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Main direction

Parallel hierarchies in:

$$\textbf{TGr} \quad \leftrightarrows \quad \textbf{DS} \quad \leftrightarrows \quad \textbf{Ban}$$

{topological groups}, {dynamical systems} and {Banach spaces}

1. **representations** of continuous group actions
   $G \curvearrowright X$ on $\mathrm{Iso}\,(V) \curvearrowright (B_{V^*}, w^*)$

   for some (nonrandom) classes $\mathcal{K}$ of Banach spaces $(V, ||\cdot||)$
   {reflexive} $\subset$ {Asplund} $\subset$ {Rosenthal} $\subset$ **Ban**

   DLP (in **Ban**) $\leftrightarrows$ WAP (in **DS**)
   fragmentability (in **Ban**) $\leftrightarrows$ HNS (in **DS**)
   Rosenthal's dichotomy (in **Ban**) $\leftrightarrows$ tameness (in **DS**)

2. **(new)** continuous actions by group automorphisms
   (e.g., actions by conjugations) and their representations
   $\downarrow$
3. counterexamples for coset G-spaces G/H.

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Banach spaces, induced structures and representations

To every $V \in \mathbf{Ban}$ one may associate:

• compact space B_{V^*} (w^*-compact unit ball in V^*)
• topological group $\mathrm{Iso}\,(V) = \{$linear onto isometries$\}$ with SOT
• dynamical system $G \times B_{V^*} \to B_{V^*}$, $(gm)(v) = m(g^{-1}v)$
for every continuous $h\colon G \to \mathrm{Iso}\,(V)$

Def: representations of actions on Banach spaces

$$\begin{array}{ccccc} G & \times & X & \longrightarrow & X \\ \downarrow h & & \downarrow \alpha & & \downarrow \alpha \\ \mathrm{Iso}\,(V) & \times & V^* & \longrightarrow & V^* \end{array}$$

Remark: [Teleman's thm]

Every compact G-space K admits a proper (embedding) representation on $V := C(K)$ as $K \hookrightarrow B_{V^*}, x \mapsto \delta_x$, $h : G \to \mathrm{Iso}\,(V), (gf)(x) = f(g^{-1}x)$.

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Question: Which actions are representable on

$$\textbf{Ref} \subset \textbf{Asp} \subset \textbf{Ros} \subset \textbf{Ban}$$

Ref = reflexive.
Asp = Asplund (V is Asplund iff every separable subspace of V is separable).
Ros = Rosenthal (a Banach space is *Rosenthal* iff $l_1 \not\subseteq V$ iff any bounded sequence contains a weak Cauchy subsequence).

$$\textbf{WAP} \subset \textbf{HNS} \subset \textbf{Tame} \subset \textbf{DS}$$

Definitions: A compact G-space X is said to be:

(a) WAP if fG is relatively weakly compact in $C(X)$ $\forall f \in C(X)$.
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Tameness of DS and Rosenthal representability

Def: A sequence $\{f_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ of functions on a set X is **independent** if $\exists\ a < b$ s.t.

$$\bigcap_{n\in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n\in M} f_n^{-1}(b, \infty) \neq \emptyset$$

for all finite disjoint subsets P, M of $\mathbb{N}$.

Fact (Glasner-Me Trans. AMS 12)

For a compact metric G-space X TFAE:

1. *G-space X is* **Ros^r**.
2. *G-space X is tame.*
3. *Ellis semigroup $E(X)$ is* **Frechet** $(scl(A) = cl(A)\ \ \forall A \subset E(X))$.
4. $card(E(X)) \leq 2^{\aleph_0}$.

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Main results of this talk

• For $G := \mathrm{SL}_n(\mathbb{R})$ the conjugation G-space G_c
is not $\mathbf{Ref^r}$ $\forall n \geq 2$;
is not $\mathbf{Asp^r}$ $\forall n \geq 4$;

• $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n$ is not $\mathbf{Asp^r}$ $\forall n \geq 3$.

• (Glasner-Me, Trans. AMS 22) $\mathrm{GL}_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$ is $\mathbf{Ros^r}$.

• For every $n \geq 2$ there exists a topological group automorphism
$\sigma : \mathbb{T}^n \to \mathbb{T}^n$ s.t.
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Two corollaries for coset spaces

• Let $G := \mathbb{T}^2 \rtimes \mathbb{Z}$ be as above. Then for its cocompact discrete subgroup $H := \mathbb{Z}$, the compact two dimensional homogeneous G-space G/H is not $\mathbf{Ros^r}$.

• There exists a closed subgroup H of $G := \mathrm{SL}_2(\mathbb{R})$ such that the corresponding locally compact coset G-space G/H is not $\mathbf{Ref^r}$.

Lemma

Let $G \curvearrowright X$ be a continuous action by group automorphisms and $P := X \rtimes_\alpha G$ be the corresponding topological semidirect product. Then $G \curvearrowright X$ naturally is embedded into the homogeneous action $P \curvearrowright P/G$.

Some questions

Question

- *For which interesting topological groups G the conjugation action $G \curvearrowright G_c$ is* $\mathbf{Ros}^{\mathbf{r}}$*?*
- *When the action $G \curvearrowright X = (G_c \cup \{\infty\})$ on the 1-point compactification is tame ?*
- *What about $G = \mathrm{SL}_2(\mathbb{R})$?*

Question

- *Which interesting homogeneous G-spaces G/H are* $\mathbf{Ros}^{\mathbf{r}}$*?*
- *What about the $\mathrm{SL}_n(\mathbb{R})$-spaces $\mathrm{SL}_n(\mathbb{R})/H$?*
- *In particular, what if $H = \mathrm{SL}_n(\mathbb{Z})$?*

More remarks about tame actions

$$\underbrace{\{\textit{Reflexive}\} \subset \{\textit{Asplund}\} \subset \{\textit{Rosenthal}\}}_{\text{"small" Banach spaces}} \subset \{\text{Banach sp.}\}$$

$$\underbrace{\{\textit{Eberlein}\} \subset \{\textit{RN}\} \subset \{\textit{WRN}\}}_{\text{"small" comp. spaces}} \subset \{\text{Compact sp.}\}$$

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Remarks about tame actions

1. Dynamical BFT (Bourgain-Fremlin Talagrand) dichotomy, Rosenthal's l_1-dichotomy [Köhler95], [GM06], [Kerr-Li07]
2. Many dynamical models in quasicrystals (tilings) are tame [Aujogue15], [Aujogue-Kellendonk15], ...
3. Closely related to NIP formulas in model theory [Shelah]
4. (Based on [Ellis 95], which in turn follows Furstenberg's results) Projective actions of $\mathrm{GL}_n(\mathbb{R})$ on the projective space $\mathbb{P}^{n-1}$ are tame but not HNS (i.e., $\mathbf{Ros^r}$ but not $\mathbf{Asp^r}$).
5. [Gl-Me] Sturmian like systems $X \subset \{0,1\}^{\mathbb{Z}^k}$ are tame. (more generally) Circularly (e.g., linearly) ordered DS
6. Every continuous G-action on a dendron is tame
7. Bernoulli cascade $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ is not tame (is not $\mathbf{Ros^r}$)

Some ingredients and details. WAP and reflexivity

Ref $=$ reflexive
$V \in$ **Ban** is reflexive ($i\colon V \hookrightarrow V^{**}$ is onto) iff for every bounded subsets $B \subset V, \ K \subset V^*$ the pairing $B \times K \to \mathbb{R}$ has Grothendieck's DLP
(for every sequence $\{f_n\} \subset B$ and every sequence $\{x_m\} \subset K$ the limits

$$\lim_n \lim_m f_n(x_m) \quad \text{and} \quad \lim_m \lim_n f_n(x_m)$$

are equal whenever they both exist)

Lemma
Let X be a compact G-space and $f \in C(X)$. TFAE:

1. $f \in \mathrm{WAP}(X)$ (fG is relatively weakly compact in $C(X)$);
2. $fG \times X \to \mathbb{R}$ has DLP;
3. fG is reflexively representable.

Thm: [Me 03] A compact metric G-space X is **Ref**$^{\mathbf{r}}$ iff X is WAP.

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$\mathrm{SL}_n(\mathbb{R}) \notin \mathbf{Ref}^r_{conj}$ $n \geq 1$

Proposition

Let $G := \mathrm{SL}_n(\mathbb{R}), n > 1$. *Then the conjugation* G*-space* G_c *is not* $\mathbf{Ref^r}$.

Sketch

We claim that the 1-point G-compactification $G_c \cup \{\infty\}$ is not WAP. It is enough to show that for every compact nbd U of $e \in G$ and for every continuous bounded function $f\colon G \to \mathbb{R}$ with $f(e) = 1$ and $f(x) = 0$ for every $x \notin U$, we have $f \notin \mathrm{WAP}(G_c)$. By **Grothendieck's double limit criterion** (for G-spaces), it suffices to show that there exist two sequences $g_n \in G$ and $x_m \in G_c$ such that the double sequence $f(g_n x_m g_n^{-1})$ $(n, m \in \mathbb{N})$ has distinct double limits.

$$g_n := \begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}, \quad x_m := \begin{pmatrix} 1 & m^{-1} \\ 0 & 1 \end{pmatrix}.$$

$$g_n x_m g_n^{-1} = \begin{pmatrix} 1 & \frac{n^2}{m} \\ 0 & 1 \end{pmatrix}$$

$$\lim_m \lim_n (g_n x_m g_n^{-1}) = \infty \quad \neq \quad \lim_n \lim_m (g_n x_m g_n^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\lim_m \lim_n f(g_n x_m g_n^{-1}) = 0 \quad \neq \quad 1 = \lim_n \lim_m f(g_n x_m g_n^{-1}).$$

$\mathrm{SL}_n(\mathbb{R}) \notin \mathbf{Ref}^r_{conj}$ $n \geq 1$

Proposition

Let $G := \mathrm{SL}_n(\mathbb{R}), n > 1$. Then the conjugation G-space G_c is not $\mathbf{Ref^r}$.

Sketch

We claim that the 1-point G-compactification $G_c \cup \{\infty\}$ is not WAP. It is enough to show that for every compact nbd U of $e \in G$ and for every continuous bounded function $f : G \to \mathbb{R}$ with $f(e) = 1$ and $f(x) = 0$ for every $x \notin U$, we have $f \notin \mathrm{WAP}(G_c)$. By **Grothendieck's double limit criterion** (for G-spaces), it suffices to show that there exist two sequences $g_n \in G$ and $x_m \in G_c$ such that the double sequence $f(g_n x_m g_n^{-1})$ $(n, m \in \mathbb{N})$ has distinct double limits.

$$g_n := \begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}, \quad x_m := \begin{pmatrix} 1 & m^{-1} \\ 0 & 1 \end{pmatrix}.$$

$$g_n x_m g_n^{-1} = \begin{pmatrix} 1 & \frac{n^2}{m} \\ 0 & 1 \end{pmatrix}$$

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Proposition

(This idea was suggested by V. Pestov) Let G be a metrizable separable topological group which is $\mathbf{Ref}^r_{conj}$. Then G is SIN (i.e., left uniformity = right uniformity)

Asplund spaces and HNS dynamical systems

• $V \in$**Asp** iff dual of every separable subspace of V is separable.
Equiv.: (Namioka-Phelps, Jane-Rogers) V_{B^*} is $(w^*, norm)$-**fragmented**
($\forall$ nonempty $A \subset V_{B^*}$ and every $\varepsilon > 0$ $\exists$ weak-star open $O \subset V^*$ s.t. $O \cap A$ is nonempty and ε-small).

• Recall the classical concept of non-sensitivity. An action of G on (X, d) is said to be *non-sensitive* if for every $\varepsilon > 0$ there exists a nonempty open subset O in X such that gO is ε-small for every $g \in G$.

• *hereditarily non-sensitive* (HNS) means that every (closed) G-subspace Y of X is non-sensitive.

Facts:
(Gl-Me 06) A compact metric G-space X is $\mathbf{Asp^r}$ iff X is HNS

(Gl-Me-Uspenskij 08) iff $E(X)$ is metrizable.

Theorem: $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n$ is not Asplund representable $\forall n \geq 3$.
The conjugation action of $\mathrm{SL}_n(\mathbb{R})$ is not **Asp^r** $\forall n \geq 4$.

Sketch:

• (S.G. Dani and S. Raghavan, Israel J. Math. 80)
$\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n$ is weakly mixing for every $n \geq 3$.

• (Glasner-Me 06) Let (X, d) be a weakly mixing G-space which is nonsensitive with respect to d. Then X is trivial.

• Now Theorem 17 implies that this action is not **Asp^r**.

• $\mathbb{R}^n \rtimes \mathrm{SL}_n(\mathbb{Z}) \hookrightarrow \mathrm{SL}_{n+1}(\mathbb{R}), \quad M \mapsto \begin{pmatrix} M & v \\ 0 & 1 \end{pmatrix}$

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Rosenthal representability, a counterexample

Theorem

For every $n \geq 2$ there exists a topological group automorphism $\sigma \colon \mathbb{T}^n \to \mathbb{T}^n$ s.t. the action of the cyclic group $\mathbb{Z}$ on $\mathbb{T}^n$ by the iterations of σ is not $\mathbf{Ros^r}$.

Proof.

For every hyperbolic toral automorphism, the corresponding cascade has positive entropy. Hence, it cannot be tame by a result of [Kerr-Li 07]. Therefore, such a cascade is not $\mathbf{Ros^r}$ by the representation thm [Glasner-Me 12]. □

Corollary

For $G = \mathbb{T}^2 \rtimes \mathbb{Z}$ its conjugation action is not $\mathbf{Ros^r}$.

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Thank you!