Pointwise attractors which are not strict

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Iterated Function System and its attractors

Definition

An **Iterated Function System (IFS)** is a pair $(X, \mathcal{F})$ where
X is a Hausdorff space
$\mathcal{F}$ is a finite family of continuous maps $X \to X$.

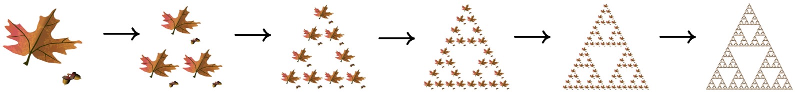
Definition

The **Barnsley-Hutchinson operator** associated with the IFS $(X, \mathcal{F})$:

$$\mathcal{F} \colon \mathcal{K}(X) \to \mathcal{K}(X) \qquad \mathcal{F}(S) = \bigcup_{f \in \mathcal{F}} f(S)$$

$\mathcal{K}(X)$ - the hyperspace of nonempty compact subsets of X with the Vietoris topology.

Strict and pointwise attractor



The set $A \in \mathcal{K}(X)$

$$\lim_{n\to\infty} \mathcal{F}^n(S) = A$$

$$\lim_{n\to\infty} \mathcal{F}^n(x) = A$$

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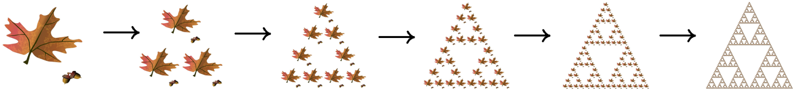

The set $A \in \mathcal{K}(X)$ is a

- **strict attractor** if there is some open nbh U of A s.t.

$$\text{for every } S \in \mathcal{K}(U),\ \lim_{n\to\infty} \mathcal{F}^n(S) = A$$

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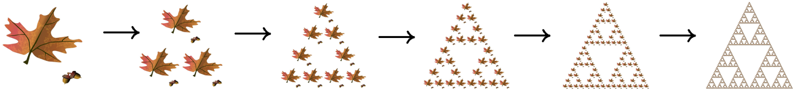

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- **strict attractor** if there is some open nbh U of A s.t.

$$\text{for every } S \in \mathcal{K}(U), \ \lim_{n \to \infty} \mathcal{F}^n(S) = A$$

- **pointwise attractor** if $A \subset \operatorname{int} B_p(A, \mathcal{F})$, where

$$B_p(A, \mathcal{F}) = \{x \in X; \ \lim_{n \to \infty} \mathcal{F}^n(x) = A\}$$

is called a pointwise basin of A

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Michael F. Barnsley, Krzysztof Leśniak, Miroslav Rypka, *Chaos game for IFSs on topological spaces* Journal of Mathematical Analysis and Applications, 435(2), 2016, 1458-1466,

Kwietniak's counterexample

$$\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$$

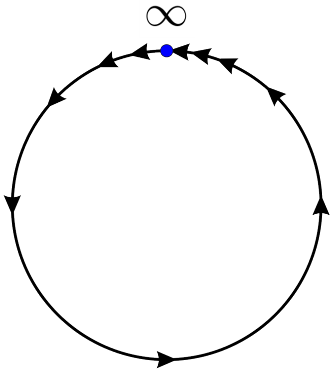
$$\varphi\colon \hat{\mathbb{R}} \to \hat{\mathbb{R}}$$

$$\varphi(x) = x + 1$$

$$\varphi(\infty) = \infty$$

$$A = \{\infty\}$$

$$B_p(A, \{\varphi\}) = \hat{\mathbb{R}}$$



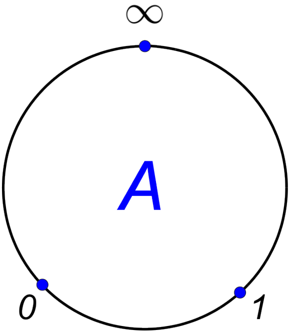
Finite union of singletons and intervals in $\mathbb{R}$ and $\hat{\mathbb{R}}$

$$\varphi\colon \hat{\mathbb{R}} \to \hat{\mathbb{R}}$$

$$\varphi(x) = \begin{cases} \dots & \text{for } x \in [\infty, 0] \\ \sqrt{x} & \text{for } x \in [0, 1] \\ \dots & \text{for } x \in [1, \infty]. \end{cases}$$

$$A = \{\infty, 0, 1\}$$

$\{\varphi, const_\infty, const_0, const_1\}$



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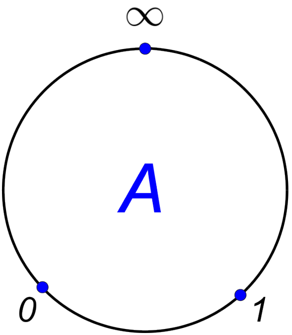
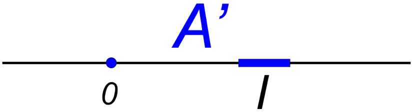


$$A' = \{0\} \cup I$$

I - retract of $\mathbb{R}$ (retraction r), pointwise attr. of $w_1, \dots, w_n$

$$\{\varphi, const_0, w_1 \circ r, \dots, w_n \circ r\}$$



A local repellor

Definition

We say that the point $a \in \mathrm{Fix}(\varphi)$ is a **local repellor** of continuous map $\varphi\colon X \to X$ on a Hausdorff space X if it has a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to a such that $x_0 \notin \mathrm{Fix}(\varphi)$ and $\varphi(x_{n+1}) = x_n$ for every $n \in \mathbb{N}$.

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Lemma (LR)

φ has a local repellor

$$\Downarrow$$

$\mathrm{Fix}(\varphi)$ is not a strict attractor for any $\mathcal{F}$ contains φ.

An attracting map

Definition

A continuous map $\varphi\colon X \to X$ on a Hausdorff space X is called an **attracting map** when
for every $x \in X$ there exists $\lim_{n\to\infty} \varphi^n(x) \in \mathrm{Fix}(\varphi)$.

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Lemma (A)

φ is an attracting map
$\mathrm{Fix}(\varphi)$ is a pointwise attractor for IFS (X, W) s.t $\mathrm{im}\, W \subset \mathrm{Fix}(\varphi)$

$$\Downarrow$$

$\mathrm{Fix}(\varphi)$ is a pointwise attractor for $(X, W \cup \{\varphi\})$.

An ALR map

Corollary $\Leftarrow$ (A) & (LR)

φ is an attracting map with local repellor
Fix(φ) is a pointwise attractor for IFS (X, W) s.t $\mathrm{im} W \subset$ Fix(φ)

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Fix(φ) is a pointwise nonstrict attractor for $(X, W \cup \{\varphi\})$.

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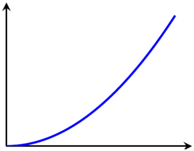
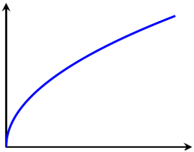


$\varphi(x) = \frac{(x-a)^2}{b-a} + a$



$\varphi(x) = \sqrt{(x-a)(b-a)} + a$

Finite sum of retracts and pointwise attractors

Theorem

φ is an attracting map with local repellor
$\mathrm{Fix}(\varphi) = \bigcup_{k=1}^{n} A_k$ where
A_k - retract of X (retraction r_k), pointwise attractor of (X, W_k)

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$$\mathcal{F} = \bigcup_{k=1}^{n} \{w \circ r_k\}_{w \in W_k} \cup \{\varphi\}.$$

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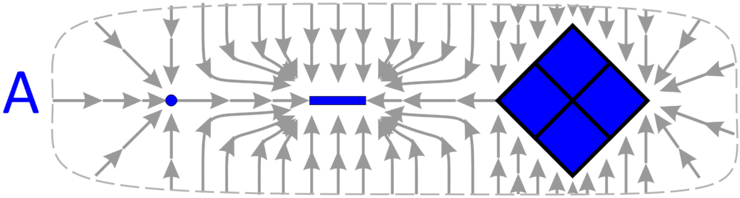
Examples: A is a finite union of

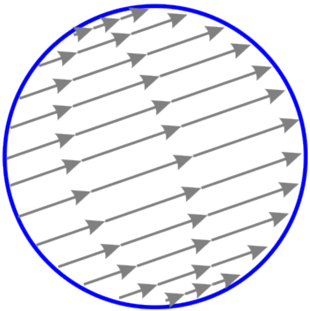
- (at least two) singletons or closed intervals in $\hat{\mathbb{R}}$ (or $\mathbb{R}$)
- (at least two) singletons, curves or sets homeo. with $[0, 1]^2$ in $\hat{\mathbb{C}}$ (or $\mathbb{C}$)

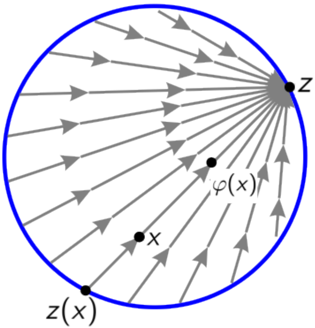
Examples

A is a finite union of at least two singletons, curves or sets homeo. with $[0,1]^2$ in $\mathbb{C}$



ALR maps on the unit ball





For a given $z \in S^1$ define $z\colon B(0,1) \to S^1$

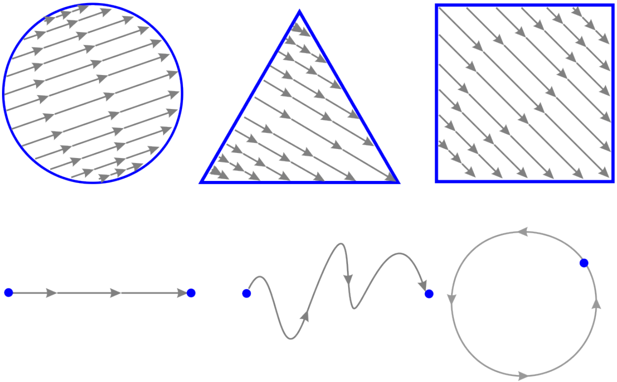
$$z(x) = \begin{cases} x & \text{for } x \in S^1 \\ e^{i \cdot arg(x-z)} & \text{for } x \notin S^1 \end{cases}$$

$$\varphi(x) = \begin{cases} z & \text{for } x = z \\ \left|\frac{x-z}{z(x)-z}\right|(x-z)+z & \text{for } x \neq z \end{cases}$$

ALR maps

Lemma

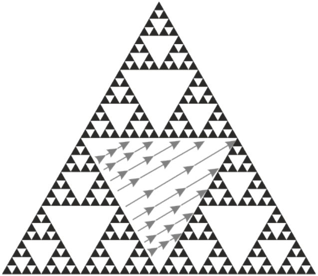
For an ALR map φ on the topological space X and for homeomorphism $h\colon X \to Y$, the composition $h \circ \varphi \circ h^{-1}$ is also an ALR map on Y.

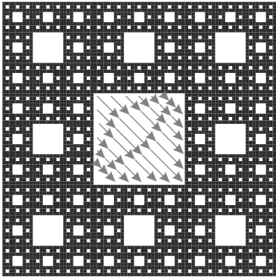


Classical fractals

Example

The Cantor set, Sierpinski triagle and Sierpiński carpet can be a pointwise attractor and no strict attractor.





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Magdalena Nowak, *Pointwise attractors which are not strict*, preprint on arXiv: http://arxiv.org/abs/2206.03244

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Slava Ukraini!