Exercises for Ordinary Differential Equations

Easy tasks (for warming up):

1) Solve the following differential equations and classify them:

•$y' = 1 + y^2$

• $y' = ay(b - y)$

• $tx\dot{x} = 1$

• $y' = xy$

• $y' = 1 - y^2$

• $x^2y' + y = 0$

2) Find the general solution of the following differential equations:

•$y''' + 3y'' + 3y' + y = 0$

• $y'''' + 4y''' + 6y'' + 4y' + y = 0$

• $y'''' + 4y''' + 3y'' - 4y' - 4y = 0.$

3) Solve the intial Value problem:

$y'' + 3y' + 4y = 0, \;\; y(0) = 1, y'(0) = 0.$

Medium tasks:

4) Check if the following differential equations are exact. Solve them if they are exact, and if not find an integrating factor and solve them.

•$x^2 + y^2 + (2 + 2xy)y' = 0$

• $y\cos(x) + (2(\sin(x) + \sin(y)) + y\cos(y))y' = 0$

• $x^3 - 3xy^2 + 2 - (3x^2 + y - y^2)y' = 0$

5) Solve the following initial value problems:

•$y' + x^2 = e^x, \;\; y(0) = 4$

• $y' - 7y + 4x = 3x^2, \;\; y(1) = 0$

• $y'' + -2y' + 3x - 7 = -\sin(x), \;\; y'(2) = 3, y(0) = 3$

6) Find solutions of the following equations:

•$\dot{x} + \frac{x}{t+1} = t^2 + t$

• $\dot{x} + \sin(t)x = \cos(t)\sin(t)$

• $4y' + 7x^3 + e^{-x^2} = \tan(x)$

7) Determine if the following functions are lipschitz-continous:

•$f : [-\frac{\pi}{2}; \frac{\pi}{2}] \to [-1, 1], f(x) = \sin(x)$

• $f : [-1, 1] \to [-\frac{\pi}{2}; \frac{\pi}{2}], f(x) = \arcsin(x)$

• $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 3x$

• $f : [-5, 3] \to [1, 9], f(x) = \frac{x^4}{x+5}$

• $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}, f(x) = \tan(x)$

• $f : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2}), f(x) = \arctan(x)$

8) What does the Picard-Lindelöf theorem say?

9) What is the use of the Gronwall-inequality?

10) Write down Liouville's formula

Hard tasks:

10) Solve the following systems of differential equations:

•$\dot{x} = Ax$, with $A = \begin{pmatrix} 7 & 1 \\ 0 & 7 \end{pmatrix}$, $x_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

• $\dot{x} = Ax$, with $A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$, $x_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

11) Find the principal matrix of the following system:

$A(t) = \begin{pmatrix} 2t & 7t^2 \\ 0 & t \end{pmatrix}$

12)Solve the following system of differential equations, find out at least one (or more) of their equilibria points and determine their class and stability:

•$\dot{x} = 4x - y \quad \dot{y} = 4 + x^2$

• $\dot{x} = y - x + 2 \quad \dot{y} = 7x + 4$

• $\dot{x} = \xi x^2 + y, \quad \dot{y} = -\eta y - x, \xi, \;\; \eta \geq 0$

13)Are the following functions lyapunov/strict lyapunov for their corresponding systems? If yes use them to find out the stability of their equilibria points:

•$L(x, y) = x^3 + y^3, \quad \dot{x} = x - y, \quad \dot{y} = -\xi x + y$

• $L(y, z) = 4z - 3y, \quad \dot{y} = z^2, \quad \dot{z} = -y^2 + z$

• $L(x, y, z) = 3x + y - 7z, \quad \dot{x} = -y, \quad \dot{y} = -yz + x, \quad \dot{z} = -3x + y$

Challenge Task:

14) Solve the following differential equation (Wolfram-Alpha won't help)

$4y^3 + 7x\sin(x) + 4x^4 - (-\frac{7}{4}y^2 + 16\cos(y) - 7x)y' = -e^{-x+y} + \sinh(y)$

Elite Task (for those who want some pocket money):

15) Looking at a fluid which is incompressible we have the following equation:

$$\nabla \cdot v = 0,$$

where v is a vector. We also have the following equation:

$$\rho(\frac{\partial v}{\partial t} + ()\nabla \cdot v)v) = -\nabla p + \mu \Delta v + f,$$

, where p is the pressure, and f the voluminaforce.

The task is to look at this equation in the three-dimensional case and study the existence and regularity of her solutions. (Have fun :P)