On entropies in quasi-metric spaces

Olivier Olela Otafudu
School of Mathematical Sciences
North-West University (Potch Campus)
Joint work with Paulus Haihambo

July 21, 2022

Introduction

The notion of entropy first appeared in the context of thermodynamics in 1864 by Rudolf Clausius as a measure of disorder/randomness of the system. In mathematics, the notion of entropy was first introduced by Shannon in the 1930's in the context of information theory.

In 1965 Adler, Konheim and McAndrew defined topological entropy of a continuous self-map of a compact metric space.

In 1971 Bowen extended this notion to uniformly continuous self-maps of (not necessarily compact) metric spaces via separating and spanning sets, which we shall call uniform entropy or Bowen's entropy.

In 1998 Kimura proposed a modified notion of entropy for uniform spaces and he proved among other things that completing a totally bounded uniform space does not change the entropy of the space, in the sense that if ψ is uniformly continuous self-map of a totally bounded uniform space $(X,\mathcal{U})$ and $(\widetilde{X},\widetilde{\mathcal{U}})$ is the completion of $(X,\mathcal{U})$ and $\widetilde{\psi}$ is the uniformly continuous extension of ψ, then the entropy of ψ and that of $\widetilde{\psi}$ coincide.

Introduction

It is well known that uniform continuity and continuity play an important role in the study of entropy in metric spaces.

Observe that for any two quasi-metric spaces (X, q) and (Y, p), if $\psi : (X, q) \to (Y, p)$ is uniformly continuous, then $\psi : (X, q^s) \to (Y, p^s)$ is uniformly continuous, but the converse is not true in general (Olela Otafudu, 2021).

As mighty be expected these have led to possibilities of studying the notion of entropy in quasi-pseudometric (quasi-metric) spaces.

Introduction

More recently (2015) Sayyari, Molari and Moghayer defined the notion of entropy for a continuous self-map of a quasi-metric space (X, q). It must be noted that they defined the entropy via join-compact subsets. i.e their notion of entropy is computed within the metric space (X, q^s).

This led to a conjecture of studying the notion of entropy of a (uniformly) continuous self-map of a quasi-metric space (X, q) using the topology $\tau(q)$ induced on X by the quasi-metric q.

Quasi-uniform entropy on a quasi-metric space

Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous map. For $x \in X$, $n \in \mathbb{N}_+$ and $\epsilon > 0$. Then we define

$$D_n^q(x, \epsilon, \psi) := \bigcap_{k=0}^{n-1} \psi^{-k}(B_q(\psi^k(x), \epsilon))$$

and

$$D_n^{q^s}(x, \epsilon, \psi) := D_n^q(x, \epsilon, \psi) \cap D_n^{q^t}(x, \epsilon, \psi).$$

It follows that

$$D_n^{q^s}(x, \epsilon, \psi) \subseteq D_n^q(x, \epsilon, \psi) \tag{1}$$

$$D_n^{q^s}(x, \epsilon, \psi) \subseteq D_n^{q^t}(x, \epsilon, \psi). \tag{2}$$

Let $\mathcal{K}(X)$ denotes the collection of all nonempty compact subsets of X. i.e. compact with respect to the topology $\tau(q)$. We define

$$r_n(\epsilon, K, \psi) = \min\left\{|F| : F \subseteq X \text{ and } K \subseteq \bigcup_{x\in F} D_n^q(x, \epsilon, \psi)\right\}$$

whenever $K \in \mathcal{K}(X)$.

A subset F of X is said to be *(n, ϵ)-supseparated* with respect to ψ if $D_n^{q^s}(x, \epsilon, \psi) \cap D_n^{q^s}(y, \epsilon, \psi) = \emptyset$ for any $x, y \in F$ with $x \neq y$.

For $K \in \mathcal{K}(X)$, we set

$$s_n(\epsilon, K, \psi) = \max\{|F| : F \subseteq K \text{ and } F \text{ is } (n, \epsilon)-\text{supseparated with respect to } \psi\}.$$

Observe that since K is compact, then the quantities $r_n(\epsilon, K, \psi)$ and $s_n(\epsilon, K, \psi)$ are finite and well defined.

Furthermore, we define:

$$r(\epsilon, K, \psi) = \lim_{n\to\infty} \sup \frac{\log r_n(\epsilon, K, \psi)}{n}$$

and

$$s(\epsilon, K, \psi) = \lim_{n\to\infty} \sup \frac{\log s_n(\epsilon, K, \psi)}{n}.$$

Then, the quantities $h_r(K, \psi)$ and $h_s(K, \psi)$ are defined by

$$h_r(K, \psi) = \lim_{\epsilon\to 0} r(\epsilon, K, \psi) \text{ and } h_s(K, \psi) = \lim_{\epsilon\to 0} s(\epsilon, K, \psi).$$

We write $r_n(\epsilon, K, \psi, q), s_n(\epsilon, K, \psi, q), r(\epsilon, K, \psi, q), s(\epsilon, K, \psi, q), h_r(K, \psi, q)$ and $h_s(K, \psi, q)$ if we need to emphasise on the quasi-metric q used.

Lemma

Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous map. If $F \subseteq K$ such that $s_n(\epsilon, K, \psi) = |F|$, then $K \subseteq \bigcup_{x \in F} D_n^{q^s}(x, \epsilon, \psi)$ whenever $K \in \mathcal{K}(X)$ and $\epsilon > 0$.

Lemma (compare Bowen)

Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous map. For any $\epsilon > 0$ and $\epsilon' > 0$ we have:

- (i) $r_n(\epsilon, K, \psi) \leq s_n(\epsilon, K, \psi) \leq r_n(\frac{\epsilon}{2}, K, \psi)$.
- (ii) If $0 < \epsilon < \epsilon'$, then
  $r_n(\epsilon, K, \psi) \geq r_n(\epsilon', K, \psi)$ and $s_n(\epsilon, K, \psi) \geq s_n(\epsilon', K, \psi)$.

Definition

Let (X, q) be a quasi-metric space, $\psi : (X, q) \to (X, q)$ be a uniformly continuous map and $K \in \mathcal{K}(X)$.

$$h_{QU}(K, \psi) = h_r(K, \psi) = h_s(K, \psi),$$

is the quasi-uniform entropy of ψ with respect to K. Furthermore, we define the quasi-uniform entropy $h_{QU}(\psi)$ of ψ by

$$h_{QU}(\psi) = \sup_{K \in \mathcal{K}(X)} h_{QU}(K, \psi).$$

We write $h_{QU}(K, \psi, q)$ and $h_{QU}(\psi, q)$ if we need to emphasise on the quasi-metric q used.

Example

Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a nonexpansive map. Then $h_{QU}(\psi) = 0$.

Proof. Suppose that $\psi : (X, q) \to (X, q)$ is a nonexpansive map. Then ψ is uniformly continuous. We first show that for $\epsilon > 0$ and $x \in X$ we have $D_n^q(x, \epsilon, \psi) = B_q(x, \epsilon)$ whenever $n \in \mathbb{N}_+$.
Let $y \in D_n^q(x, \epsilon, \psi)$. It follows that $q(\psi^k(x), \psi^k(y) < \epsilon$ for any $0 \leq k < n$. By taking $k = 0$, we have $y \in B_q(x, \epsilon)$.
If $y \in B_q(x, \epsilon)$, then $q(\psi^k(x), \psi^k(y)) \leq q(x, y) < \epsilon$ since ψ is nonexpansive. Hence $y \in D_n^q(x, \epsilon, \psi)$. We have that $r_n(\epsilon, K, \psi)$ and $s_n(\epsilon, K, \psi)$ do not depend on $n \in \mathbb{N}_+$. Thus, $r(\epsilon, K, \psi) = 0 = s(\epsilon, K, \psi)$.
Therefore, $h_{QU}(\psi) = 0$. $\square$

Definition (compare Bowen)

Two quasi-metrics q_1 and q_2 on a set X are uniformly equivalent if $id_X : (X, q_1) \to (X, q_2)$ and $id_X : (X, q_2) \to (X, q_1)$ are both uniformly continuous maps of quasi-metric spaces. In this case $\psi : (X, q_1) \to (X, q_1)$ is uniformly continuous if and only if $\psi : (X, q_2) \to (X, q_2)$ is uniformly continuous.

If (X,q) is a quasi-metric space, then (X, q) and (X, q^t) are not uniformly equivalent in general.

For instance consider $\mathbb{R}$, equipped with the usual quasi-metric u, defined by $u(x, y) = \max\{x - y, 0\}$ for all $x, y \in \mathbb{R}$.
Then $id_X : (X, u) \to (X, u^t)$ is not uniformly continuous, as it is not continuous. In particular $id_X : (X, u) \to (X, u^t)$ is not continuous on $(-\infty, 0]$.

Indeed, for any $\delta > 0$ choose $x < 0$ and set $\epsilon = \frac{|x|}{2} > 0$, then
$u(x, 0) = \max\{x, 0\} = 0 < \delta$, but
$u^t(id_X(x), id_X(0)) = u^t(x, 0) = u(0, x) = \max\{-x, 0\} = -x = |x| > \epsilon$.

Therefore $id_X : (X, u) \to (X, u^t)$ is not continuous at 0, and so is not uniformly continuous.

Lemma

If q_1 and q_2 are uniformly equivalent quasi-metrics on X and $\psi : (X, q_1) \to (X, q_1)$ is uniformly continuous, then $h_{QU}(\psi, q_1) = h_{QU}(\psi, q_2)$.

Lemma

(i) *Let* (X,q) *be a quasi-metric space and* $\psi : (X,q) \to (X,q)$ *be a uniformly continuous map, then* $h_{QU}(\psi^m) = mh_{QU}(\psi)$ *for each* $m \in \mathbb{N} = \mathbb{N}_+ \cup \{0\}$.

(ii) *Let* (X_1,q_1) *and* (X_2,q_2) *be quasi-metric spaces . Suppose* $\psi_1 : (X_1,q_1) \to (X_1,q_1)$ *and* $\psi_2 : (X_2,q_2) \to (X_2,q_2)$ *are uniformly continuous maps. Define a quasi-metric* q *on* $X_1 \times X_2$ *by*

$$q((x_1,x_2),(y_1,y_2) = \max\{q_1(x_1,y_1), q_2(x_2,y_2)\},$$

for all $(x_1,x_2),(y_1,y_2) \in X_1 \times X_2$.
Then $\psi_1 \times \psi_2 : (X_1 \times X_2, q) \to (X_1 \times X_2, q)$ *is uniformly continuous and* $h_{QU}(\psi_1 \times \psi_2, q) \leq h_{QU}(\psi_1,q_1) + h_{QU}(\psi_2,q_2)$. *Furthermore, if* X_1 *or* X_2 *is compact, then* $h_{QU}(\psi_1 \times \psi_2, q) = h_{QU}(\psi_1,q_1) + h_{QU}(\psi_2,q_2)$.

Lemma

Let (X, q) be a quasi-metric space. Then

(i) $\psi : (X, q) \to (X, q)$ *is uniformly continuous if and only if* $\psi : (X, q^t) \to (X, q^t)$ *is uniformly continuous.*

(ii) *if* $\psi : (X, q) \to (X, q)$ *is uniformly continuous, then* $\psi : (X, q^s) \to (X, q^s)$ *is uniformly continuous. The converse does not hold in general.*

Definition (Bowen)

Let (X,d) be a metric space and $\psi:(X,d)\to(X,d)$ be a uniformly continuous function. For each $x\in X$, $n\in\mathbb{N}_+$ and $\epsilon>0$. Then we define the uniform entropy h_U of ψ by

$$h_U(\psi)=\sup\{h_U(K,\psi): K \text{ is a compact subset of } X\},$$

where $h_U(K,\psi)$ is defined in exactly the same way as in previous section. We write $h_U(K,\psi,d)$ and $h_U(\psi,d)$ to emphasise the metric d used.

Lemma

Let (X,q) be a quasi-metric space and $\psi:(X,q)\to(X,q)$ be a uniformly continuous function. Let $n\in\mathbb{N}_+$ and $\epsilon>0$.

(i) *If $K\subseteq X$ is join-compact and $F\subseteq X$ such that $K\subseteq\bigcup_{x\in F}D_n^{q^s}(x,\epsilon,\psi)$, then $K\subseteq\bigcup_{x\in F}D_n^{q}(x,\epsilon,\psi)$.*

(ii) *$F\subseteq X$ is (n,ϵ)-separated if and only if $F\subseteq X$ is (n,ϵ)-supseparated.*

Theorem

Let (X, q) be a quasi-metric space and $\psi : (X, q) \to (X, q)$ be a uniformly continuous function. Then

(i) $h_U(\psi, q^s) \geq h_{QU}(\psi, q)$.

(ii) $h_U(\psi, q^s) \geq \min\{h_{QU}(\psi, q), h_{QU}(\psi, q^t)\}$

Example

Let us consider $\mathbb{R}$ equipped with the usual quasi-metric u ($u(x, y) = \max\{x - y, 0\}$ for all $x, y \in \mathbb{R}$).
Let $\psi : (\mathbb{R}, u) \to (\mathbb{R}, u)$ be the map defined by $\psi(x) = 2x$ for each $x \in \mathbb{R}$.
Then ψ is uniformly continuous and

$$0 = h_{QU}(\psi, u) \leq h_U(\psi, u^s) = \log 2.$$

Completion theorem for quasi-uniform entropy on a quasi-metric space

Definition

Let (X, q) be a quasi-metric space and $\psi : X \to X$ a function. A subset Y of X is *ψ-invariant if* $\psi(Y) \subseteq Y$.

If (X, q) is a quasi-metric space and $Y \subseteq X$. It is well known that (Y, q_Y) is a quasi-metric space, where $q_Y = q|_{Y \times Y}$.

Lemma

Let (X, q) be a quasi-metric space, $\psi : (X, q) \to (X, q)$ a uniformly continuous function and let Y be an ψ-invariant subset of X. For $n \in \mathbb{N}_+$ and $\epsilon > 0$ we have that

(i) *If $y \in Y$, then $D_n^{q_Y}(y, \epsilon, \psi|_Y) = D_n^q(y, \epsilon, \psi) \cap Y$ and $D_n^{(q_Y)^s}(y, \epsilon, \psi|_Y) = D_n^{q^s}(y, \epsilon, \psi) \cap Y$.*

(ii) *$r_n(\epsilon, K, \psi|_Y, q_Y) = r_n(\epsilon, K, \psi, q)$ for each $K \in \mathcal{K}(Y)$*

Lemma

Let (X, q) be a quasi-metric space, $\psi : (X, q) \to (X, q)$ a uniformly continuous function and let Y be an ψ-invariant subset of X. Then $h_{QU}(\psi|_Y, q_Y) \leq h_{QU}(\psi, q)$.

Definition

Let (X, q) be a quasi-metric space.

(1) (X, q) is said to be *bicomplete* if the metric space (X, q^s) is complete. i.e every q^s-Cauchy sequence is q^s-convergent.

(2) A *bicompletion* of (X, q) is a pair $\left(\varphi, (\widetilde{X}, \widetilde{q})\right)$ consisting of a bicomplete quasi-metric space $(\widetilde{X}, \widetilde{q})$ and a quasi-isometry $\varphi : (X, q) \to (\widetilde{X}, \widetilde{q})$ such that $\varphi(X)$ is dense in the metric space $(\widetilde{X}, (\widetilde{q})^s)$.

Theorem

Let (X,q) be a quasi-metric space and $\psi : (X,q) \to (X,q)$ a uniformly continuous function. If $(\widetilde{X},\widetilde{q})$ is the bicompletion of (X,q) and $\widetilde{\psi}$ is the uniformly continuous extension of ψ over $\widetilde{X}$. Then

$$h_{QU}(\psi,q) \leq h_{QU}(\widetilde{\psi},\widetilde{q}).$$

Theorem

Let (X,q) be a join-compact quasi-metric space and $\psi : (X,q) \to (X,q)$ a uniformly continuous function. If $(\widetilde{X},\widetilde{q})$ is the bicompletion of (X,q) and $\widetilde{\psi}$ is the uniformly continuous extension of ψ over $\widetilde{X}$. Then

$$h_{QU}(\psi,q) = h_{QU}(\widetilde{\psi},\widetilde{q}).$$

Thank you