Typical Trees: An Out(F_r) Excursion

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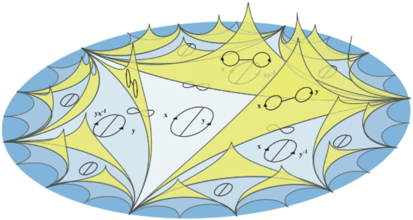

I. Main Characters (Propaganda): CV_r & $Out(F_r)$

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To define $\Phi \in Aut(F_r)$, just need to describe images of generators:

$$\Phi = \begin{cases} x_1 \mapsto x_1 x_3^{-1} \\ x_2 \mapsto x_3 \\ x_3 \mapsto x_2 \end{cases}$$

Viewing $Out(F_r)$: Generated by simple "replacement functions" (Nielsen)

For each $i \neq j$, ordering, etc, have generator:

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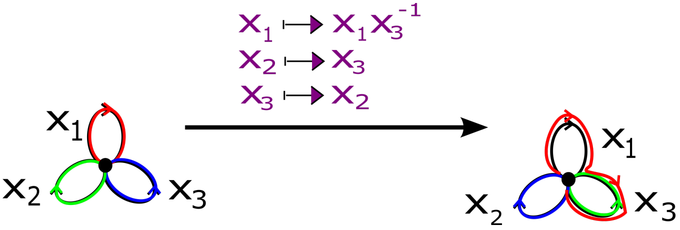
So, for φ defined by $x_2 \mapsto x_2 x_3^{-1}$ $(x_2^{-1} \mapsto x_3 x_2^{-1})$

$\varphi(x_3^{-1} x_2 x_1 x_2^{-1}) = x_3^{-1} x_2 x_3^{-1} x_1 x_3 x_2^{-1}$

Viewing $Out(F_r)$: Homotopy equivalences of graphs Γ with $\pi_1(\Gamma) \cong F_r$

Example:



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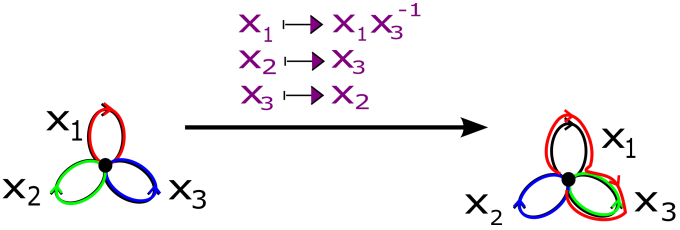
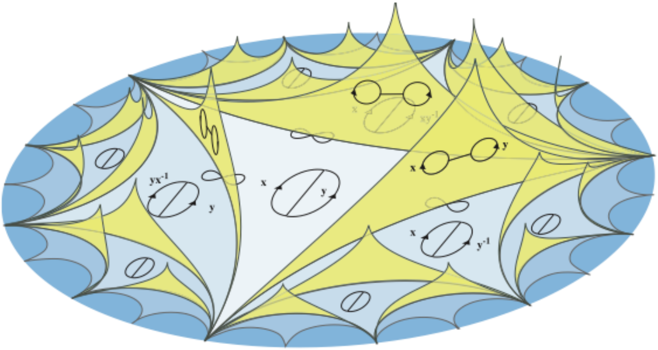


(Images of F_r generators $\leftrightsquigarrow$ Images of corresponding loops)

Viewing $Out(F_r)$: $Out(F_r) \cong Isom(CV_r)$

$Out(F_r)$ is the isometry group for a deformation space of metric graphs, Culler-Vogtmann Outer Space CV_r



II. Main Questions & Inadequate Knowledge Summary

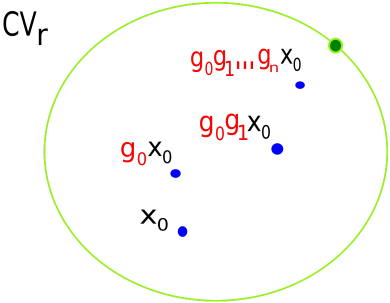
Underlying Questions

Handel-Mosher Question

$$\underbrace{g_0 g_1 \cdots g_n}\cdots$$

Stop, composing yields automorphism
What properties does it typically have?
What if compose more generators?

Bestvina Question



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typically converges to a ∂CV_r pt
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Inadequate Knowledge Summary

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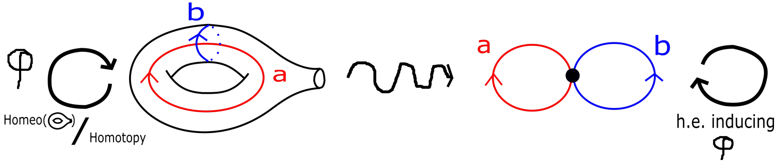
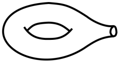
• [Rivin, '08] φ is **fully irreducible**: No positive power φ^k fixes the conjugacy class of a nontrivial proper free factor of F_r

• [Rivin, '10] φ is **NOT induced by a surface homeomorphism**:



III. Asymptotic Conjugacy Class Invariant T_+^{φ}

The Backstory

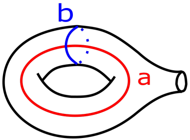
$GL_2(Z) \cong$ MCG() $\cong Out(F_2)$

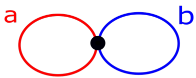
2x2 integer matrices
of determinant +/- 1

Homeo() / Homotopy

$Aut(F_2) / Inn(F_2)$

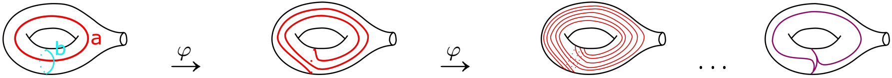
$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$





Nielsen-Thurston studied asymptotic dynamical invariants

For φ a generic surface homeo, repeated application of φ to any curve limits on the same object...



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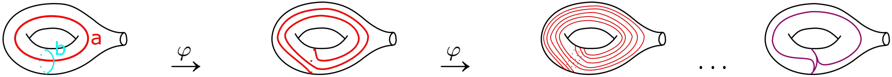

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Some important "conjugacy class" invariants:

$GL(2,\mathbb{Z})$	$MCG(\Sigma_{1,1})$	$Out(F_2)$

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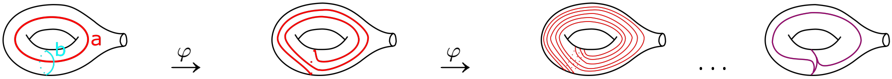

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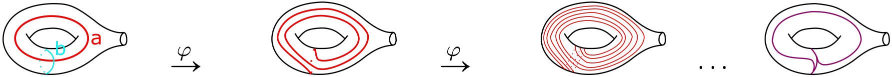

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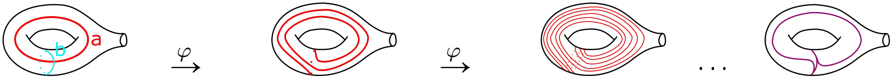

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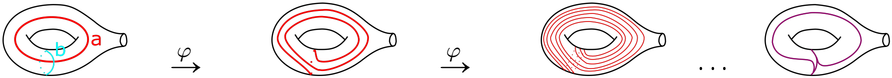

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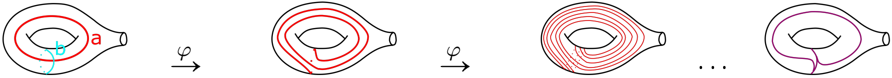

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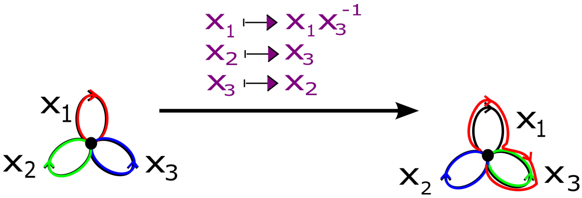
* Connected to "measured foliations"

To understand these invariants in our circumstance, we need to understand train track representatives...

Train Track Representatives (Bestvina-Handel)

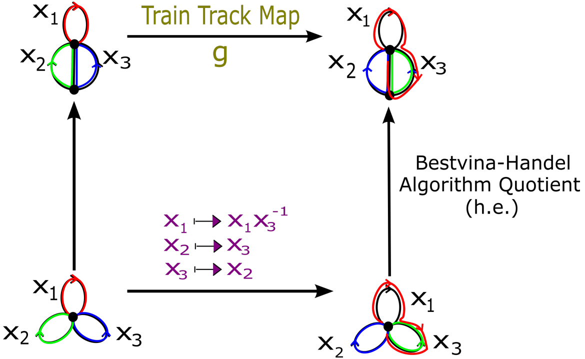
Recall: $\varphi \in Out(F_r)$ always have topological representatives:



- But iteration may lead to cancellation on edge interiors

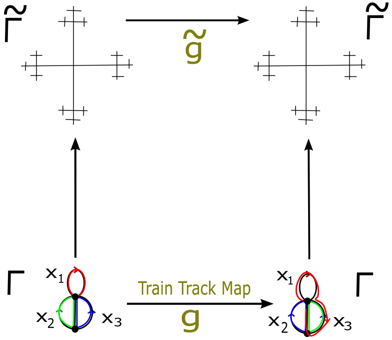
Train Track Representatives (Bestvina-Handel)

Nice $\varphi \in Out(F_r)$ have **train track representatives** $g\colon \Gamma \xrightarrow{h.e.} \Gamma$



No cancellation on edge interiors even after iteration!

Asymptotic Dynamical Invariant of Interest: Attracting Tree T_+^φ



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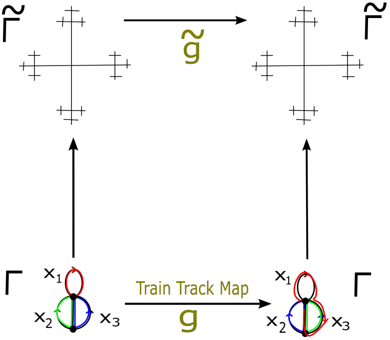

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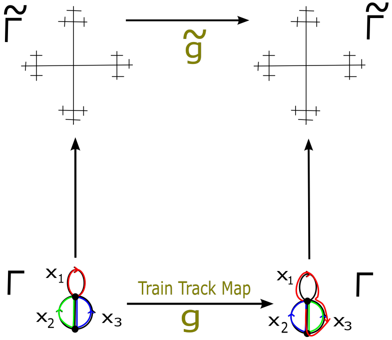

- Can assign edge-lengths so that g stretches all by stretch factor $\lambda(\varphi)$
- T_+^φ is quotient of $\tilde{\Gamma}$ with metric

$$d_\infty(x, y) = lim_{k\to\infty}\left(\frac{d(\tilde{g}^k(x), \tilde{g}^k(y))}{\lambda^k}\right)$$

IV. Answers

Recall Underlying Questions

Handel-Mosher Question

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What properties does it typically have?
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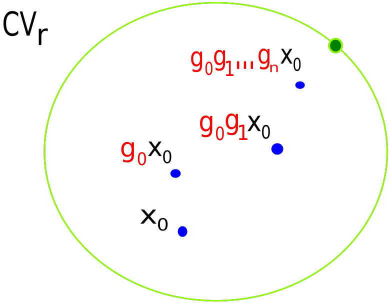

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T_+^{φ} is always an $\mathbb{R}$-**tree**, i.e. 0-hyperbolic space

$\mathbb{R}$-trees can be:

1. Simplicial trees
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3.
4.

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   - f.i.'s mimic pseudo-Anosov homeos
   - pseudo-Anosovs $\rightsquigarrow$ foliations $\rightsquigarrow$ geometric $\mathbb{R}$-trees
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4. Have k-**pronged branch points** p: $T \setminus \{p\}$ has $k \geqslant 3$ components

Main Random Walk Theorems

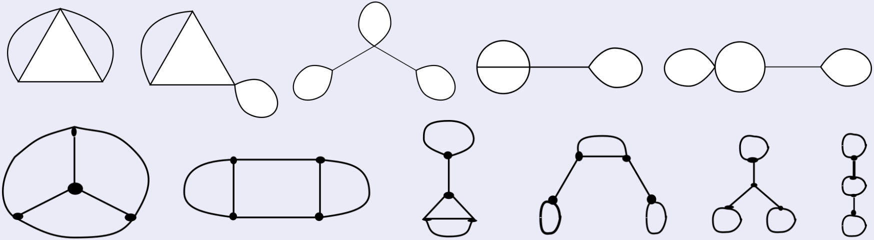
Theorem (I. Kapovich, J. Maher, C. Pfaff, S. Taylor)

IN BOTH CIRCUMSTANCES TYPICALLY GET A VERY HAIRY AGEOMETRIC $\mathbb{R}$-TREE WITH ALL BRANCH POINTS TRIVALENT!

What the outer automorphisms may look like

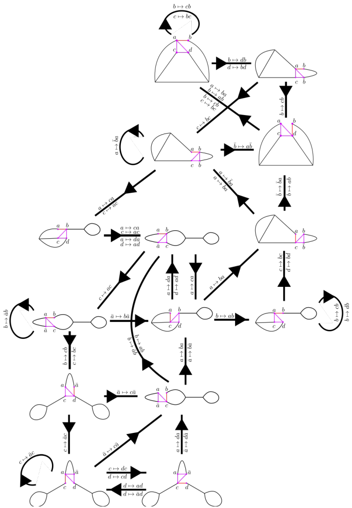
Theorem (D. Gagnier, C. Pfaff)

The graphs carrying train track representatives of "principal" fully irreducible outer automorphisms in $Out(F_3)$ are precisely:



The simplices in CV_3 with principal axes passing through them are precisely those whose these underlying graphs.

The "principal axes" automaton



Thank you!