The mean orbital pseudo-metric in topological dynamics

Habibeh Pourmand
joint work with Fangzhou Cai, Dominik Kwietniak, Jian Li

JAGIELLONIAN UNIVERSITY IN KRAKOW

University of Vienna, 19 July 2022

Iceland Liechtenstein Norway grants
Norway grants

Outline

- Introduction
- Notation
- The mean orbital pseudo-metric and invariant measures
- $\bar{\rho}$-continuity and mean equicontinuity

Introduction

Let (X, T) be a **topological dynamical system**, that is

Introduction

Let (X, T) be a **topological dynamical system**, that is

- X is a compact metric space endowed with a metric d;

Introduction

Let (X, T) be a **topological dynamical system**, that is

- X is a compact metric space endowed with a metric d;
- $T\colon X \to X$ is a continuous map.

Introduction

Let (X, T) be a **topological dynamical system**, that is

- X is a compact metric space endowed with a metric d;
- $T\colon X \to X$ is a continuous map.

The **orbit** of x is:

$$\mathrm{Orb}(x, T) = \{T^n(x) : n \geqslant 0\}.$$

Introduction

The **Besicovitch pseudo-metric**

$$\rho_B(x, y) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k(x), T^k(y)),$$

where $x, y \in X$.

Introduction

The **mean orbital pseudo-metric**

$$\bar{\rho}(x, y) = \limsup_{n\to\infty} \min_{\sigma\in S_n} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k(x), T^{\sigma(k)}(y)),$$

where $x, y \in X$ and S_n is the permutation group on $\{0, 1, \ldots, n-1\}$.

Introduction

The **mean orbital pseudo-metric**

$$\bar{\rho}(x, y) = \limsup_{n\to\infty} \min_{\sigma\in S_n} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k(x), T^{\sigma(k)}(y)),$$

where $x, y \in X$ and S_n is the permutation group on $\{0, 1, \ldots, n-1\}$.

- L. Zheng and Z. Zheng 2020

Introduction

The **mean orbital pseudo-metric**

$$\bar{\rho}(x,y) = \limsup_{n\to\infty} \min_{\sigma\in S_n} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k(x), T^{\sigma(k)}(y)),$$

where $x, y \in X$ and S_n is the permutation group on $\{0, 1, \ldots, n-1\}$.

- L. Zheng and Z. Zheng 2020
- $\bar{\rho} \leqslant \rho_B$

Notations

Let (X, T) be a topological dynamical system.

Notations

Let (X, T) be a topological dynamical system.

- $\boldsymbol{\mathcal{M}(X)}$ is the space of Borel probability measures on X.

Notations

Let (X, T) be a topological dynamical system.

- $\boldsymbol{\mathcal{M}(X)}$ is the space of Borel probability measures on X.
- **Prokhorov metric** for $\mu, \nu \in \mathcal{M}(X)$

$$\rho(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(B) \leqslant \nu(B^{\varepsilon}) + \varepsilon \text{ for every Borel set } B \subset X\},$$

Notations

Let (X, T) be a topological dynamical system.

- $\boldsymbol{\mathcal{M}(X)}$ is the space of Borel probability measures on X.
- **Prokhorov metric** for $\mu, \nu \in \mathcal{M}(X)$

$$\rho(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(B) \leqslant \nu(B^\varepsilon) + \varepsilon \text{ for every Borel set } B \subset X\},$$

- **Hausdorff metric** for $A, B \in 2^{\mathcal{M}(X)}$

$$\rho_H(A, B) = \max\big\{\inf\{\varepsilon > 0 : B \subset A^\varepsilon\}, \inf\{\varepsilon > 0 : A \subset B^\varepsilon\}\big\}.$$

Notations

Let (X, T) be a topological dynamical system.

- $\boldsymbol{\mathcal{M}(X)}$ is the space of Borel probability measures on X.
- **Prokhorov metric** for $\mu, \nu \in \mathcal{M}(X)$

  $$\rho(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(B) \leqslant \nu(B^\varepsilon) + \varepsilon \text{ for every Borel set } B \subset X\},$$

- **Hausdorff metric** for $A, B \in 2^{\mathcal{M}(X)}$

  $$\rho_H(A, B) = \max\big\{\inf\{\varepsilon > 0 : B \subset A^\varepsilon\}, \inf\{\varepsilon > 0 : A \subset B^\varepsilon\}\big\}.$$

Notations

Let (X, T) be a topological dynamical system.

- $\boldsymbol{\mathcal{M}(X)}$ is the space of Borel probability measures on X.
- **Prokhorov metric** for $\mu, \nu \in \mathcal{M}(X)$

$$\rho(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(B) \leqslant \nu(B^{\varepsilon}) + \varepsilon \text{ for every Borel set } B \subset X\},$$

- **Hausdorff metric** for $A, B \in 2^{\mathcal{M}(X)}$

$$\rho_H(A, B) = \max\big\{\inf\{\varepsilon > 0 : B \subset A^{\varepsilon}\}, \inf\{\varepsilon > 0 : A \subset B^{\varepsilon}\}\big\}.$$

Remark

$$\rho_H(\{\mu\}, \{\nu\}) = \rho(\mu, \nu) \quad \textit{for every } \mu, \nu \in \mathcal{M}(X).$$

Notation

Given $x \in X$ and $n \in \mathbb{N}$,

Notation

Given $x \in X$ and $n \in \mathbb{N}$,

- $\hat{\delta}(T^n(x))$ is the **Dirac measure** concentrated at $T^n(x)$.

Notation

Given $x \in X$ and $n \in \mathbb{N}$,

- $\hat{\delta}(T^n(x))$ is the **Dirac measure** concentrated at $T^n(x)$.
- **Empirical measure**

$$m_T(x, n) = \frac{1}{n} \sum_{i=0}^{n-1} \hat{\delta}(T^i(x)).$$

Notation

Given $x \in X$ and $n \in \mathbb{N}$,

- $\hat{\delta}(T^n(x))$ is the **Dirac measure** concentrated at $T^n(x)$.
- **Empirical measure**

$$m_T(x, n) = \frac{1}{n} \sum_{i=0}^{n-1} \hat{\delta}(T^i(x)).$$

- $\mu \in \mathcal{M}(X)$ is a **distribution measure** of $x \in X$ if μ is a limit of some subsequence of $\{m_T(x, n)\}_{n=1}^{\infty}$.

Notation

Given $x \in X$ and $n \in \mathbb{N}$,

- $\hat{\delta}(T^n(x))$ is the **Dirac measure** concentrated at $T^n(x)$.
- **Empirical measure**

$$m_T(x,n) = \frac{1}{n}\sum_{i=0}^{n-1} \hat{\delta}(T^i(x)).$$

- $\mu \in \mathcal{M}(X)$ is a **distribution measure** of $x \in X$ if μ is a limit of some subsequence of $\{m_T(x,n)\}_{n=1}^{\infty}$.
- $\boldsymbol{\hat{\omega}(x)}$ is the set of all distribution measures of $x \in X$.

Notation

Given $x \in X$ and $n \in \mathbb{N}$,

- $\hat{\delta}(T^n(x))$ is the **Dirac measure** concentrated at $T^n(x)$.
- **Empirical measure**

$$m_T(x,n) = \frac{1}{n} \sum_{i=0}^{n-1} \hat{\delta}(T^i(x)).$$

- $\mu \in \mathcal{M}(X)$ is a **distribution measure** of $x \in X$ if μ is a limit of some subsequence of $\{m_T(x,n)\}_{n=1}^{\infty}$.
- $\boldsymbol{\hat{\omega}(x)}$ is the set of all distribution measures of $x \in X$.
- $\hat{\omega}(x)$ is a non-empty closed connected subset of $\mathcal{M}_T(X)$.

Notation

Given $x \in X$ and $n \in \mathbb{N}$,

- $\hat{\delta}(T^n(x))$ is the **Dirac measure** concentrated at $T^n(x)$.
- **Empirical measure**

$$m_T(x,n) = \frac{1}{n}\sum_{i=0}^{n-1}\hat{\delta}(T^i(x)).$$

- $\mu \in \mathcal{M}(X)$ is a **distribution measure** of $x \in X$ if μ is a limit of some subsequence of $\{m_T(x,n)\}_{n=1}^{\infty}$.
- $\boldsymbol{\hat{\omega}(x)}$ is the set of all distribution measures of $x \in X$.
- $\hat{\omega}(x)$ is a non-empty closed connected subset of $\mathcal{M}_T(X)$.
- $x \in X$ is a **generic point** for $\mu \in \mathcal{M}_T(X)$ if $\hat{\omega}(x) = \{\mu\}$.

The mean orbital pseudo-metric and invariant measures

Definition

A TDS (X, T) is mean equicontinuous if for $\varepsilon > 0$ there exists δ such that for every $x, y \in X$ with $d(x, y) < \delta$ one has $\rho_B(x, y) < \varepsilon$.

The mean orbital pseudo-metric and invariant measures

Definition
A TDS (X, T) is mean equicontinuous if for $\varepsilon > 0$ there exists δ such that for every $x, y \in X$ with $d(x, y) < \delta$ one has $\rho_B(x, y) < \varepsilon$.

Definition
A TDS (X, T) is $\bar{\rho} -$ continuous if for every $\varepsilon > 0$ there exists δ such that for every $x, y \in X$ with $d(x, y) < \delta$ one has $\bar{\rho}(x, y) < \varepsilon$.

The mean orbital pseudo-metric and invariant measures

Definition

A TDS (X, T) is mean equicontinuous if for $\varepsilon > 0$ there exists δ such that for every $x, y \in X$ with $d(x, y) < \delta$ one has $\rho_B(x, y) < \varepsilon$.

Definition

A TDS (X, T) is $\bar{\rho} -$ continuous if for every $\varepsilon > 0$ there exists δ such that for every $x, y \in X$ with $d(x, y) < \delta$ one has $\bar{\rho}(x, y) < \varepsilon$.

- Mean equicontinuty and $\bar{\rho}$-continuity do not depend on the metric d on X which induce the same topology.

The mean orbital pseudo-metric and invariant measures

Definition
A TDS (X, T) is mean equicontinuous if for $\varepsilon > 0$ there exists δ such that for every $x, y \in X$ with $d(x, y) < \delta$ one has $\rho_B(x, y) < \varepsilon$.

Definition
A TDS (X, T) is $\bar{\rho} -$ continuous if for every $\varepsilon > 0$ there exists δ such that for every $x, y \in X$ with $d(x, y) < \delta$ one has $\bar{\rho}(x, y) < \varepsilon$.

- Mean equicontinuty and $\bar{\rho}$-continuity do not depend on the metric d on X which induce the same topology.
- Mean equicontinuity is equivalent to **stability in the mean in the sense of Lyapunov** introduced by Fomin.

The mean orbital pseudo-metric and invariant measures

Theorem (CKLP '22+)

For a TDS (X, T)*, the map* $(X, \bar{\rho}) \to (2^{\mathcal{M}_T(X)}, \rho_H)$*,* $x \to \hat{\omega}(x)$ *is uniformly continuous, that is for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for every* $x, y \in X$ *with* $\bar{\rho}(x, y) < \delta$ *one has* $\rho_H(\hat{\omega}(x), \hat{\omega}(y)) < \varepsilon$*.*

The mean orbital pseudo-metric and invariant measures

Theorem (CKLP '22+)

For a TDS (X, T), the map $(X, \bar{\rho}) \to (2^{\mathcal{M}_T(X)}, \rho_H)$, $x \to \hat{\omega}(x)$ is uniformly continuous, that is for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $x, y \in X$ with $\bar{\rho}(x, y) < \delta$ one has $\rho_H(\hat{\omega}(x), \hat{\omega}(y)) < \varepsilon$.

- D. Kwietniak, M. Łącka, and P. Oprocha 2017

The mean orbital pseudo-metric and invariant measures

Theorem (CKLP '22+)

For a TDS (X, T)*, the map* $(X, \bar{\rho}) \to (2^{\mathcal{M}_T(X)}, \rho_H)$*,* $x \to \hat{\omega}(x)$ *is uniformly continuous, that is for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for every* $x, y \in X$ *with* $\bar{\rho}(x, y) < \delta$ *one has* $\rho_H(\hat{\omega}(x), \hat{\omega}(y)) < \varepsilon$*.*

- D. Kwietniak, M. Łącka, and P. Oprocha 2017

Corollary (CKLP '22+)

Let (X, T) *be a TDS. For any* $x, y \in X$ *if* $\bar{\rho}(x, y) = 0$ *then* $\hat{\omega}(x) = \hat{\omega}(y)$*.*

The mean orbital pseudo-metric and invariant measures

Let (X, T) be a TDS. Define

$$\bar{\rho}_n(x, y) = \min_{\sigma \in S_n} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k x, T^{\sigma(k)} y), \ \ n \in \mathbb{N}.$$

Recall $\bar{\rho}(x, y) = \limsup_{n\to\infty} \bar{\rho}_n(x, y)$.

The mean orbital pseudo-metric and invariant measures

Let (X, T) be a TDS. Define

$$\bar{\rho}_n(x, y) = \min_{\sigma \in S_n} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k x, T^{\sigma(k)} y), \; n \in \mathbb{N}.$$

Recall $\bar{\rho}(x, y) = \limsup_{n\to\infty} \bar{\rho}_n(x, y)$.

Theorem (CKLP '22+)

Let (X, T) be a TDS. If x is a generic point, then for every $\varepsilon > 0$, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that for every $y \in X$ and $n \geqslant N$ with $\rho(m_T(x, n), m_T(y, n)) < \delta$, one has $\bar{\rho}_n(x, y) < \varepsilon$.

The mean orbital pseudo-metric and invariant measures

Let (X, T) be a TDS. Define

$$\bar{\rho}_n(x, y) = \min_{\sigma \in S_n} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k x, T^{\sigma(k)} y), \;\; n \in \mathbb{N}.$$

Recall $\bar{\rho}(x, y) = \limsup_{n\to\infty} \bar{\rho}_n(x, y)$.

Theorem (CKLP '22+)

Let (X, T) be a TDS. If x is a generic point, then for every $\varepsilon > 0$, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that for every $y \in X$ and $n \geqslant N$ with $\rho(m_T(x, n), m_T(y, n)) < \delta$, one has $\bar{\rho}_n(x, y) < \varepsilon$.

Corollary (CKLP '22+)

Let (X, T) be a TDS. If x is a generic point, then for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $y \in X$ with $\rho_H(\hat{\omega}(x), \hat{\omega}(y)) < \delta$, one has $\bar{\rho}(x, y) < \varepsilon$.

The mean orbital pseudo-metric and invariant measures

Corollary (Zheng, Zheng '20)

Let (X, T) be a TDS. If $x \in X$ is a generic point, then for every $y \in X$, $\hat{\omega}(x) = \hat{\omega}(y)$ if and only if $\bar{\rho}(x, y) = 0$.

The mean orbital pseudo-metric and invariant measures

Corollary (Zheng, Zheng '20)

Let (X, T) be a TDS. If $x \in X$ is a generic point, then for every $y \in X$, $\hat{\omega}(x) = \hat{\omega}(y)$ if and only if $\bar{\rho}(x, y) = 0$.

Corollary (Zheng, Zheng '20)

A TDS (X, T) is uniquely ergodic if and only if $\bar{\rho}(x, y) = 0$ for all $x, y \in X$.

$\bar{\rho}$-continuity and mean equicontinuity

Lemma (CKLP '22+)

If (X, T) is a TDS such that the map $\hat{\omega}\colon (X, d) \to (2^{\mathcal{M}_T(X)}, \rho_H)$, $x \to \hat{\omega}(x)$ is continuous, then for every $x \in X$ the TDS $(\overline{\mathrm{Orb}(x, T)}, T)$ is uniquely ergodic.

$\bar{\rho}$-continuity and mean equicontinuity

Lemma (CKLP '22+)

If (X,T) is a TDS such that the map $\hat{\omega}\colon (X,d) \to (2^{\mathcal{M}_T(X)}, \rho_H)$, $x \to \hat{\omega}(x)$ is continuous, then for every $x \in X$ the TDS $(\overline{\mathrm{Orb}(x,T)}, T)$ is uniquely ergodic.

Lemma (CKLP '22+)

Let (X,T) be a TDS. If (X,T) is $\bar{\rho}$-continuous and has a dense orbit, then (X,T) is uniquely ergodic.

Main Theorem

Theorem (CKLP '22+)

Let (X, T) be a TDS. Then the following statements are equivalent:

Main Theorem

Theorem (CKLP '22+)

Let (X, T) be a TDS. Then the following statements are equivalent:

1. (X, T) *is* $\bar{\rho}$*-continuous;*

Main Theorem

Theorem (CKLP '22+)

Let (X, T) be a TDS. Then the following statements are equivalent:

1. (X, T) *is $\bar{\rho}$-continuous;*
2. *the map $\hat{\omega}\colon (X, d) \to (2^{\mathcal{M}_T(X)}, \rho_H)$ is continuous;*

Main Theorem

Theorem (CKLP '22+)

Let (X, T) be a TDS. Then the following statements are equivalent:

1. (X, T) *is* $\bar{\rho}$*-continuous;*
2. *the map* $\hat{\omega}\colon (X, d) \to (2^{\mathcal{M}_T(X)}, \rho_H)$ *is continuous;*
3. *empirical measure maps* $(m_T(\cdot, n))_{n=1}^{\infty}$*, where* $m_T(\cdot, n)\colon X \to \mathcal{M}(X)$ *and*

$$x \mapsto m_T(x, n) = \frac{1}{n} \sum_{j=0}^{n-1} \hat{\delta}(T^j(x)) \qquad \text{for } x \in X \text{ and } n \in \mathbb{N},$$

are uniformly equicontinuous on X*;*

$\bar{\rho}$-continuity and mean equicontinuity

4. every $x \in X$ is generic for some ergodic invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T^e(X)$ is continuous;

$\bar{\rho}$-continuity and mean equicontinuity

4. every $x \in X$ is generic for some ergodic invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}^e_T(X)$ is continuous;
5. every $x \in X$ is generic for some invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T(X)$ is continuous;

$\bar{\rho}$-continuity and mean equicontinuity

4. every $x \in X$ is generic for some ergodic invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T^e(X)$ is continuous;
5. every $x \in X$ is generic for some invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T(X)$ is continuous;
6. for every continuous function $f\colon X \to \mathbb{R}$, the sequence of continuous functions $\{\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\}$ is pointwise convergent to a continuous function f^*;

$\bar{\rho}$-continuity and mean equicontinuity

4. every $x \in X$ is generic for some ergodic invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T^e(X)$ is continuous;
5. every $x \in X$ is generic for some invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T(X)$ is continuous;
6. for every continuous function $f\colon X \to \mathbb{R}$, the sequence of continuous functions $\{\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\}$ is pointwise convergent to a continuous function f^*;
7. for every continuous function $f\colon X \to \mathbb{R}$, the sequence of continuous functions $\{\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\}$ converges uniformly.

$\bar{\rho}$-continuity and mean equicontinuity

4. every $x \in X$ is generic for some ergodic invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T^e(X)$ is continuous;
5. every $x \in X$ is generic for some invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T(X)$ is continuous;
6. for every continuous function $f\colon X \to \mathbb{R}$, the sequence of continuous functions $\{\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\}$ is pointwise convergent to a continuous function f^*;
7. for every continuous function $f\colon X \to \mathbb{R}$, the sequence of continuous functions $\{\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\}$ converges uniformly.

If any of the conditions 1–7 holds, then the limit continuous function f^* mentioned in 6 is the function given for $x \in X$ by

$$f^*(x) = \int_X f \,\mathrm{d}\mu(x).$$

$\bar{\rho}$-continuity and mean equicontinuity

4. every $x \in X$ is generic for some ergodic invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T^e(X)$ is continuous;
5. every $x \in X$ is generic for some invariant measure and the map $X \ni x \mapsto \mu(x) \in \mathcal{M}_T(X)$ is continuous;
6. for every continuous function $f\colon X \to \mathbb{R}$, the sequence of continuous functions $\{\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\}$ is pointwise convergent to a continuous function f^*;
7. for every continuous function $f\colon X \to \mathbb{R}$, the sequence of continuous functions $\{\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\}$ converges uniformly.

If any of the conditions 1–7 holds, then the limit continuous function f^* mentioned in 6 is the function given for $x \in X$ by

$$f^*(x) = \int_X f \,\mathrm{d}\mu(x).$$

- T. Downarowicz, B. Weiss, L. Zheng, Z. Zheng, $\cdots$

$\bar{\rho}$-continuity and mean equicontinuity

Definition

We say that (X, T) is $\{\bar{\rho}_n\}$ − equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for every $x, y \in X$ with $d(x, y) < \delta$ for every $n \in \mathbb{N}$ we have $\bar{\rho}_n(x, y) < \varepsilon$.

$\bar{\rho}$-continuity and mean equicontinuity

Definition

We say that (X, T) is $\{\bar{\rho}_n\}$ − equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for every $x, y \in X$ with $d(x, y) < \delta$ for every $n \in \mathbb{N}$ we have $\bar{\rho}_n(x, y) < \varepsilon$.

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

$\bar{\rho}$-continuity and mean equicontinuity

Definition

We say that (X, T) is $\{\bar{\rho}_n\}$ − equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for every $x, y \in X$ with $d(x, y) < \delta$ for every $n \in \mathbb{N}$ we have $\bar{\rho}_n(x, y) < \varepsilon$.

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

1. (X, T) *is $\bar{\rho}$-continuous;*

$\bar{\rho}$-continuity and mean equicontinuity

Definition

We say that (X, T) is $\{\bar{\rho}_n\}$ − equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for every $x, y \in X$ with $d(x, y) < \delta$ for every $n \in \mathbb{N}$ we have $\bar{\rho}_n(x, y) < \varepsilon$.

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

1. *(X, T) is $\bar{\rho}$-continuous;*
2. *(X, T) is $\{\bar{\rho}_n\}$-equicontinuous.*

$\bar{\rho}$-continuity and mean equicontinuity

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

$\bar{\rho}$-continuity and mean equicontinuity

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

1. (X, T) *is mean equicontinuous;*

$\bar{\rho}$-continuity and mean equicontinuity

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

1. (X, T) *is mean equicontinuous;*
2. $(X \times X, T \times T)$ *is $\bar{\rho}$-continuous;*

$\bar{\rho}$-continuity and mean equicontinuity

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

1. (X, T) *is mean equicontinuous;*
2. $(X \times X, T \times T)$ *is $\bar{\rho}$-continuous;*
3. *for every $(x, y) \in X \times X$ the system $\overline{\mathrm{Orb}((x, y), T \times T)}$ is uniquely ergodic and the map $(x, y) \mapsto \mu(x, y)$ is continuous;*

$\bar{\rho}$-continuity and mean equicontinuity

Theorem (CKLP '22+)

For a TDS (X, T) *the following conditions are equivalent:*

1. (X, T) *is mean equicontinuous;*
2. $(X \times X, T \times T)$ *is* $\bar{\rho}$*-continuous;*
3. *for every* $(x, y) \in X \times X$ *the system* $\overline{\mathrm{Orb}((x, y), T \times T)}$ *is uniquely ergodic and the map* $(x, y) \mapsto \mu(x, y)$ *is continuous;*
4. (X, T) *is Weyl mean equicontinuous: if for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for every* $x, y \in X$ *with* $d(x, y) < \delta$

$$\limsup_{n-m\to\infty} \frac{1}{n-m} \sum_{k=m}^{n-1} d(T^n x, T^n y) < \varepsilon.$$

$\bar{\rho}$-continuity and mean equicontinuity

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

1. (X, T) *is mean equicontinuous;*
2. $(X \times X, T \times T)$ *is $\bar{\rho}$-continuous;*
3. *for every $(x, y) \in X \times X$ the system $\overline{\mathrm{Orb}((x, y), T \times T)}$ is uniquely ergodic and the map $(x, y) \mapsto \mu(x, y)$ is continuous;*
4. (X, T) *is Weyl mean equicontinuous: if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta$*

$$\limsup_{n-m\to\infty} \frac{1}{n-m} \sum_{k=m}^{n-1} d(T^n x, T^n y) < \varepsilon.$$

- J. Li, S. Tu, X. Ye 2015; T. Downarowicz, E. Glanser 2016

$\bar{\rho}$-continuity and mean equicontinuity

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

1. (X, T) *is mean equicontinuous;*
2. $(X \times X, T \times T)$ *is $\bar{\rho}$-continuous;*
3. *for every $(x, y) \in X \times X$ the system $\overline{\mathrm{Orb}((x, y), T \times T)}$ is uniquely ergodic and the map $(x, y) \mapsto \mu(x, y)$ is continuous;*
4. (X, T) *is Weyl mean equicontinuous: if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta$*

$$\limsup_{n-m\to\infty} \frac{1}{n-m} \sum_{k=m}^{n-1} d(T^n x, T^n y) < \varepsilon.$$

- J. Li, S. Tu, X. Ye 2015; T. Downarowicz, E. Glanser 2016
- J. Qiu, J. Zhao 2020

$\bar{\rho}$-continuity and mean equicontinuity

Theorem (CKLP '22+)

For a TDS (X, T) the following conditions are equivalent:

1. (X, T) *is mean equicontinuous;*
2. $(X \times X, T \times T)$ *is $\bar{\rho}$-continuous;*
3. *for every $(x, y) \in X \times X$ the system $\overline{\mathrm{Orb}((x,y), T \times T)}$ is uniquely ergodic and the map $(x, y) \mapsto \mu(x, y)$ is continuous;*
4. (X, T) *is Weyl mean equicontinuous: if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta$*

$$\limsup_{n-m\to\infty} \frac{1}{n-m} \sum_{k=m}^{n-1} d(T^n x, T^n y) < \varepsilon.$$

- J. Li, S. Tu, X. Ye 2015; T. Downarowicz, E. Glanser 2016
- J. Qiu, J. Zhao 2020
- G. Fuhrmann, M. Gröger, and D. Lenz 2022

References

F. Cai, D. Kwietniak, J. Li, H. Pourmand
On the properties of the mean orbital pseudo-metric
J. Differential Equations. ***318*** *(2022), 1-19. 37B05 (37A25).*

T. Downarowicz, E. Glasner
Isomorphic Extensions and Applications
Topol. Methods Nonlinear Anal. **48** (2016), no. 1, 321–338.

T. Downarowicz, B. Weiss
When all points are generic for ergodic measures
Bull. Pol. Acad. Sci. Math. **68** (2020), no. 2, 117–132.

S. Fomin
On dynamical systems with a purely point spectrum
Doklady Akad. Nauk SSSR (N.S.) **77** (1951), 29-32.

G. Fuhrmann, M. Gröger, D. Lenz
The structure of mean equicontinuous group actions
Israel J. Math. **247** (2022) , 75-123.

References

D. Kwietniak, M. Łącka, P. Oprocha
Generic points for dynamical systems with average shadowing
Monatsh. Math. **183** (2017), no. 4, 625–648.

J. Li, S. Tu, X. Ye
Mean equicontinuity and mean sensivity
Ergodic Theory Dynam. Systems. **35**(2015), no. 8, 2587-2612

J. Qiu, J. Zhao
A note on mean equicontinuity
J. Dynam. Differential Equations. **32** (2020), no. 1, 101–116.

L. Zheng, Z. Zheng
A new metric for statistical properties of long term behaviours
*J. Differential Equations. **269**(4)(2020) 2741-2773.*