Midterm Exam
ODEs Summer term

Instructions: no books, no notes, no calculators, etc. Please show all work: it is more important than the answer, and Don't Panic!

1) $\frac{dy}{dx} = (2 - y)y$

a) Determine equilibrium solutions to the above Differential Equation and state whether the above equilibrium solutions are asymptotically stable, unstable or semi-stable.

b) find y(5), if y(0)= 1

c) Sketch the graph of the direction field.
(10 points)

2) A 100 liter tank of water is initially filled with 50 liters of fresh water. Salt water which has a concentration $= 1Molar = \frac{1mole}{liter} = 1M$ is then added at the rate of 2 liters per minute. At the same time as this is being done, 1 liter of the contents of the tank is being leaked per minute. Assuming that the solution is perfectly mixed throughout the whole tank, what is the concentration of salt after 10 minutes?
(15 points)

3) Show that $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$
(5 points)

4) For the next question you DO NOT need to solve the Differential Equation, merely state the "simplest" guess that can be expected to give a solution to the following inhomogeneous Differential equations, given that:
$L[y] = y'' + 2y' + 2y$
a) $L[y] = \cos(x)(x^3 + x + 1)$
b) $L[y] = e^{-x}\cos(-x)(x^3)$
c) $L[y] = \cos(x)(x^3 + x + 1) + e^{-x}\cos(-x)(x^3 + 1)$
(10 points)

5) For the differential operator:

$$L[y] = \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y$$

Find the general solution to: $L[y] = e^x \cos(x)$
(15 points)

6) Find the general solution to:

$$x^2 y + e^{x+y} + (\frac{1}{3}x^3 + e^{x+y})y' = x \sin(x^2)$$

(10 points)

7)Solve for y, given that $y(1) = 1$, and

$$\frac{dy}{dx} + \frac{1}{x}y = e^x$$

(10 points)

8) Find a second fundamental solution to the ODE:

$$x^2 y'' - xy' + \frac{3}{4}y = 0$$

using that $y_1(x) = x^{\frac{3}{2}}$ is also a solution.
Remark: The use of a method that has been covered in this course will be important: I don't want to see any other types of solutions to this ODE. Sorry.
(10 points)

9) Find the general solution to:

$$\cos(x)y'' + \cos(x)y = \sin(x),\ x \in (-\frac{\pi}{4}, \frac{\pi}{4})$$

Hint: some integrals and trigonometric identities are included on the last part of this exam.
(15 points)

10) Find the general solution to the ODE:

$$\sin(y) + \cos(y)y' = -e^x$$

(15 points)

You may need some of the following facts, to finish this exam:

$$\int \frac{1}{\cos(x)} dx = \ln|\sec(x) + \tan(x)| + C$$

$$\sin(x)^2 + \cos(x)^2 = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

$$\cos(a+b) = -\sin(a)\sin(b) + \cos(a)\cos(b)$$

$$e^{a+ib} = e^a(\cos(b) + i\sin(b))$$