ODEs
Final Exam **(100 points)**

Instructions: Show all of your work. *No credit will be awarded for an answer without the necessary work.*

1. (**20 points**)

   (a) Solve each IVP:

   i. $x' = \frac{t}{2x(t+1)}$, $x(0) = 1$

   ii. $x' = t^3 - 2tx$, $x(0) = 1$.

   (b) Find the general solution of each ODE:

   i. $x' = \frac{t}{t^2+1}$

   ii. $x''' + x'' + x' + x = 0$

2. (**15 points**)

   (a) Here is the basic existence and uniqueness theorem:

   > Suppose $F(t,u) \in \mathbb{R}^n$ is defined for all $t \in (a,b)$ and $u \in \mathbb{R}^n$ satisfying $\alpha_i \le u_i \le \beta_i, i = 1, \dots, n$. Suppose further that F is continuous, and its derivative with respect to each u_i exists and is continuous. Then, for any t_0, $u^{(0)}$ satisfying $a < t_0 < b$, $\alpha_i < u_i^{(0)} < \beta_i$, $i = 1, \dots, n$, there exists $\epsilon > 0$ such that
   >
   > $$u' = F(t,u), \quad u(t_0) = u^{(0)}$$
   >
   > has a unique solution existing on the interval $(t_0 - \epsilon, t_0 + \epsilon)$.

   Explain how the conclusions of the theorem change if F has the form $F(t,u) = A(t)u + f(t)$, where $A : (a,b) \to \mathbb{R}^{n\times n}$ and $f : (a,b) \to \mathbb{R}^n$ are continuous.

   (b) Consider a homogeneous linear ODE $u' = A(t)u$, where $A : (a,b) \to \mathbb{R}^{n\times n}$ is continuous, and a corresponding inhomogeneous linear ODE $u' = A(t)u + f(t)$, where $f : (a,b) \to \mathbb{R}^n$ is also continuous. Let $u(t) = c_1u^{(1)} + c_2u^{(2)} + \cdots + c_nu^{(n)}$ be the general solution of the homogeneous ODE (this means that every solution of the ODE can be written uniquely in this form). What is the general solution of the inhomogeneous ODE? Prove your answer (that is, prove that every solution of the inhomogeneous ODE can be written uniquely in the form you state).

   (c) Let $a_i : \mathbb{R} \to \mathbb{R}$ be continuous for each $i = 0, 1, \dots, n-1$ and consider the ODE

   $$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_1(t)x' + a_0(t)x = 0.$$

   Suppose $x_1, x_2, \dots, x_n$ are all solutions of this ODE and every solution x of the ODE can be written in the form $x(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t)$. What must be true of the functions $x_1, x_2, \dots, x_n$? Explain (preferably using concepts of linear algebra).

3. (**20 points**) Consider the ODE $tx'' - (3+t)x' + 2x = 0$ for $t > 0$.

   (a) Show that 0 is a regular singular point of the ODE.

   (b) Find the two roots of the indicial equation.

   (c) Find a solution in the form $x_1(t) = \sum_{n=0}^{\infty} a_n t^{n+m_1}$, where m_1 is the larger of the two roots.

   (d) Is there a solution of the form $x_2(t) = \sum_{n=0}^{\infty} a_n t^{n+m_2}$, where m_2 is the smaller root? Is so, find it. If not, give the correct form for the second solution (but do not compute this solution).

4. (**20 points**) Let $A \in \mathbb{R}^{3\times 3}$ be defined by $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. The eigenvalue/eigenvector pairs of A are as follows: $\lambda_1 = 1, x^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\lambda_2 = 1, x^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\lambda_3 = -1, x^{(3)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

   (a) Find the general solution of $u' = Au$.

   (b) Find e^{tA}.

   (c) Find a formula for the solution of the IVP $u' = Au + \begin{bmatrix} \phi(t) \\ 0 \\ 0 \end{bmatrix}$, $u(0) = 0$. (Notice the special form of the right-hand side: $f(t) = (\phi(t), 0, 0)$.)

5. (**25 points**) Consider the following population model for two species:

$$u_1' = u_1(1 - u_1 + 0.5u_2),$$
$$u_2' = u_2(2.5 - 1.5u_2 + 0.25u_1).$$

   Here $u_1(t)$ and $u_2(t)$ are the populations (in suitable units, say thousands of animals) at time t of two species that share the same environment.

   (a) Find all the equilibrium points of the system.

   (b) For each equilibrium point:

   - Linearize the system around that point.
   - Classify the equilibrium point (its type and stability properties) as an equilibrium point of the *linear system.*
   - Classify the equilibrium point (its type and stability properties) as an equilibrium point of the *nonlinear system.*

   (c) Is it the case that, for all initial populations with $u_1(0) > 0$ and $u_2(0) > 0$, the trajectories all converge to the same equilibrium point? Explain.