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Stability of the topological pressure for continuous piecewise monotonic interval maps

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A map $T : [0,1] \to [0,1]$ is called a *piecewise monotonic map* if there exists a finite partition $\mathcal{Z}$ of $[0,1]$ into pairwise disjoint open intervals with $\overline{\bigcup_{Z\in\mathcal{Z}} Z} = [0,1]$ such that $T\big|_Z$ is continuous and strictly monotonic (increasing or decreasing) for every $Z \in \mathcal{Z}$. It is called C^1 if T is continuously differentiable on $(0,1)$ and T' can be extended to a continuous function on $[0,1]$.
For a C^1-map $T : [0,1] \to [0,1]$ define
$\|T\| := \max_{x\in[0,1]} |Tx| + \sup_{x\in(0,1)} |T'x|$. Moreover, we will also consider the R^0-topology on continuous piecewise monotonic maps defined below. Given $\varepsilon > 0$ two continuous piecewise monotonic maps T_1 and T_2 are said to be ε-close in the R^0-topology, if

- they have the same number of intervals of monotonicity,
- the endpoints of corresponding intervals of monotonicity differ at most by ε, and
- $\max_{x\in[0,1]} |T_1 x - T_2 X| < \varepsilon$.

Next we fix an $N \in \mathbb{N}$. Denote by $\mathcal{M}_N$ the family of all C^1-maps $S : [0,1] \to [0,1]$ which are piecewise monotonic with at most N intervals of monotonicity. The set $\mathcal{M}_N$ is endowed with the C^1-topology, this is the topology given by the norm defined above. Suppose that $T : [0,1] \to [0,1]$ is a C^1-map such that $\text{card}\big(\{c \in (0,1) : T'c = 0\}\big) \leq N-1$. Hence $\{c \in (0,1) : T'c = 0\}$ is finite and $T \in \mathcal{M}_N$.
Given $n \in \mathbb{N}$ and $\varepsilon > 0$ a subset $E \subseteq [0,1]$ is called (n,ε)-separated if for any $x \neq y \in E$ there is a $j \in \{0,1,\ldots,n-1\}$ with $|T^j x - T^j y| \geq \varepsilon$. If $f : [0,1] \to \mathbb{R}$ is continuous then the topological pressure $p(T,f)$ is defined by

$$p(T,f) := \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log \sup_E \left(\sum_{x \in E} \exp \left(\sum_{j=0}^{n-1} f\left(T^j x\right) \right) \right) ,$$

where the supremum is taken over all (n,ε)-separated subsets $E \subseteq [0,1]$. Moreover define $h_{\text{top}}(T) = p(T,0)$ the topological entropy of T.

The variational principle states that

$$p(T,f) = \sup_{\mu \in \mathcal{P}(T)} \left(h_\mu(T) + \int f \, d\mu \right) =$$
$$= \sup_{\mu \in \mathcal{E}(T)} \left(h_\mu(T) + \int f \, d\mu \right) \ ,$$

where $\mathcal{P}(T)$ is the set of all T-invariant Borel probability measures μ on $[0,1]$, $\mathcal{E}(T)$ is the set of all ergodic T-invariant Borel probability measures μ on $[0,1]$, and $h_\mu(T)$ is the measure-theoretic entropy of T. A T-invariant Borel probability measure μ on $[0,1]$ is called measure of maximal entropy if $h_\mu(T) = h_{\text{top}}(T)$.

Concerning lower semi-continuity of the topological pressure we have the following results.

Theorem 1

Suppose that $T : [0,1] \to [0,1]$ *is a piecewise monotonic map, that* $f : [0,1] \to \mathbb{R}$ *is continuous and satisfies* $p(T,f) > \lim_{n\to\infty} \frac{1}{n} \sup_{x\in[0,1]} \sum_{j=0}^{n-1} f(T^j x)$. *Then*

$$\liminf_{\substack{\widetilde{T}\to T \ in\ R^0 \\ \|\widetilde{f}-f\|_\infty \to 0}} p(\widetilde{T},\widetilde{f}) \geq p(T,f)\ .$$

Theorem 2

Let $T : [0,1] \to [0,1]$ *be a* C^1*-map such that* $\mathrm{card}\big(\{c \in (0,1) : T'c = 0\}\big) \leq N-1$. *Suppose that* $f : [0,1] \to \mathbb{R}$ *is continuous and satisfies* $p(T,f) > \lim_{n\to\infty} \frac{1}{n} \sup_{x\in[0,1]} \sum_{j=0}^{n-1} f(T^j x)$. *Then*

$$\liminf_{\substack{\|\widetilde{T}-T\|\to 0 \\ \widetilde{T}\in\mathcal{M}_N \\ \|\widetilde{f}-f\|_\infty\to 0}} p(\widetilde{T},\widetilde{f}) \geq p(T,f)\ .$$

Note that $p(T,f) > \sup_{x\in[0,1]} f(x)$ implies $p(T,f) > \lim_{n\to\infty} \frac{1}{n} \sup_{x\in[0,1]} \sum_{j=0}^{n-1} f(T^j x)$. The supremum above is always a maximum.

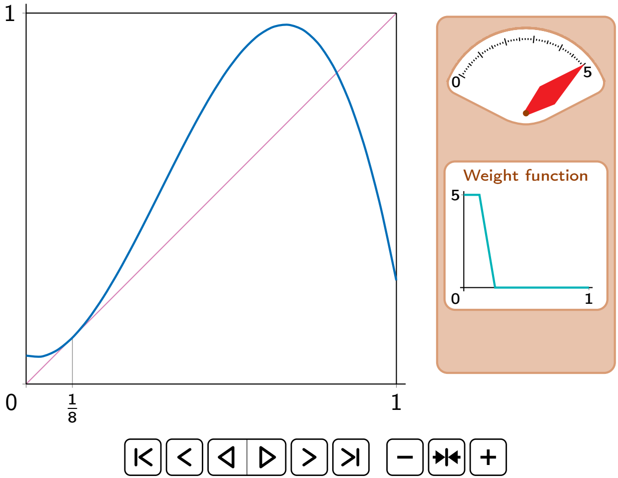
Define $T : [0,1] \to [0,1]$ by

$$Tx := \frac{29\,865}{389\,344} - \frac{3\,878}{12\,167}x + \frac{77\,530}{12\,167}x^2 - \frac{71\,200}{12\,167}x^3 \,.$$

Observe that $\frac{1}{8}$ is a fixed point of T, and that $\left[\frac{1}{4}, 1\right]$ is T-invariant. We define $f : [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 5\,, & \text{if } x \in \left[0, \frac{1}{8}\right], \\ 10 - 40x\,, & \text{if } x \in \left[\frac{1}{8}, \frac{1}{4}\right], \\ 0\,, & \text{if } x \in \left[\frac{1}{4}, 1\right]. \end{cases}$$

Moreover, for $s \in \left[0, \frac{1}{32}\right]$ set $T_s x := Tx + s$ (hence $T_0 = T$). Note that T_s converges to T for $s \to 0$ both in the R^0- and C^1-topology. Below we will see that the pressure is not lower semi-continuous in this case.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Weight function
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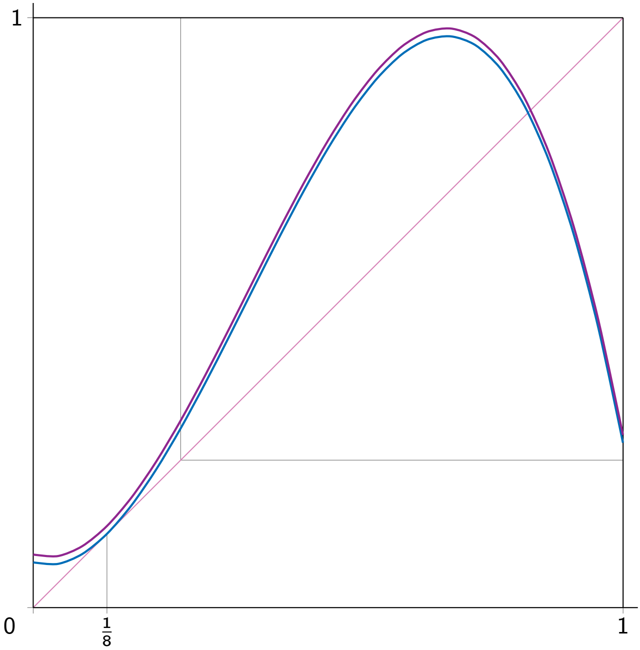
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$\frac{1}{8}$
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Since T_s is invariant on $\left[\frac{1}{4}, 1\right]$ we obtain $h_{\text{top}}(T_s) \leq \log 2$. As $\frac{1}{8}$ is a fixed point of T we get $p(T, f) = 5$. In the case $s > 0$ this fixed point is destroyed and every orbit comes into $\left[\frac{1}{4}, 1\right]$ after some time we get $p(T_s, f) = h_{\text{top}}(T_s) \leq \log 2$ showing the "jump down" of the pressure. For this example one can show that the endpoints of the intervals of monotonicity are not periodic, hence one has upper semi-continuity of the pressure by Theorem 9.

We obtain as an easy consequence of Theorem 1 and Theorem 2 that the topological entropy is lower semi-continuous. Here the condition for lower semi-continuity would be $h_{\text{top}}(T) > 0$, and lower semi-continuity is trivial in the case $h_{\text{top}}(T) = 0$ as the entropy is always nonnegative. The result on lower semi-continuity of entropy has been obtained in 1980 by Michał Misiurewicz and Wiesław Szlenk.

For C^1-maps we get the following result on upper semi-continuity of the topological pressure.

Theorem 3

Let $T : [0,1] \to [0,1]$ *be a* C^1*-map such that* $\operatorname{card}\big(\{c \in (0,1) : T'c = 0\}\big) \leq N-1$*. Suppose that* $f : [0,1] \to \mathbb{R}$ *is continuous. Then*

$$\limsup_{\substack{\|\widetilde{T}-T\|\to 0 \\ \widetilde{T}\in\mathcal{M}_N \\ \|\widetilde{f}-f\|_\infty\to 0}} p(\widetilde{T},\widetilde{f}) = p(T,f)\ .$$

Using Theorem 2 and Theorem 3 we obtain immediately the following result concerning the topological entropy which has been proved by Michał Misiurewicz in 1995.

Theorem 4

Assume that $T:[0,1]\to[0,1]$ *is a* C^1*-map such that* $\operatorname{card}\big(\{c\in(0,1): T'c=0\}\big)\le N-1$*. Then*

$$\lim_{\substack{\|\widetilde{T}-T\|\to 0\\ \widetilde{T}\in\mathcal{M}_N}} h_{\mathrm{top}}(\widetilde{T}) = h_{\mathrm{top}}(T)\ .$$

Next we deal with the measure of maximal entropy. In general a dynamical system may not have a measure of maximal entropy. Furthermore a map may have more than one measure of maximal entropy. Every dynamical system with a measure of maximal entropy has always also an ergodic measure of maximal entropy. We obtain the following result for C^1-maps.

Theorem 5

Let $T : [0,1] \to [0,1]$ *be a* C^1*-map such that* $\mathrm{card}\big(\{c \in (0,1) : T'c = 0\}\big) \leq N-1$. *Suppose that* $h_{\mathrm{top}}(T) > 0$ *and* μ *is the unique measure of maximal entropy for* T. *Then there exists a* $\delta > 0$ *such that every* $\widetilde{T} \in \mathcal{M}_N$ *with* $\|\widetilde{T} - T\| < \delta$ *has a unique measure* $\widetilde{\mu}$ *of maximal entropy. Moreover,*

$$\lim_{\substack{\|\widetilde{T}-T\|\to 0 \\ \widetilde{T}\in\mathcal{M}_N}} \widetilde{\mu} = \mu \ ,$$

where the limit is meant in the w^**-topology.*

By Theorem 4 and Theorem 5 one easily obtains that

$$\lim_{\substack{\|\widetilde{T}-T\|\to 0 \\ \widetilde{T}\in\mathcal{M}_N}} h_{\widetilde{\mu}}(\widetilde{T}) = h_{\mu}(T) \ .$$

Observe that in the situation considered above the perturbed map may have more intervals of monotonicity than the original map. It is only assumed that the number of intervals of monotonicity is bounded by a given number N. Nonetheless this number could be chosen arbitrary large.
The situation is very different if one considers piecewise monotonic maps with respect to the R^0-topology. Here the perturbed map has the same number of intervals of monotonicity as the original map. Unfortunately one does not get upper semi-continuity of the pressure and the entropy in this case. We like to describe the possible "jumps up" in this situation. Denote by E the set of all endpoints of intervals of monotonicity of T except 0 and 1. For a periodic point x denote by $n(x)$ the period of x and set

$$k(x) := \mathrm{card}\left(E \cap \left\{x, Tx, T^2x, \ldots, T^{n(x)-1}x\right\}\right) .$$

Theorem 6

Let $T : [0,1] \to [0,1]$ *be a piecewise monotonic map, and suppose that that* $f : [0,1] \to \mathbb{R}$ *is continuous. Then*

$$\limsup_{\substack{\widetilde{T} \to T \text{ in } R^0 \\ \|\widetilde{f} - f\|_\infty \to 0}} p(\widetilde{T}, \widetilde{f}) = \max\left\{ p(T, f), \max\left\{ \frac{k(x)}{n(x)} \log 2 + \right.\right.$$

$$\left.\left. + \frac{1}{n(x)} \sum_{j=0}^{n(x)-1} f\left(T^j x\right) : x \in E \text{ is periodic} \right\}\right\} .$$

As an easy consequence of Theorem 6 we get the following result on entropy which has been obtained by Michał Misiurewicz in 1989.

Theorem 7

Suppose that $T : [0,1] \to [0,1]$ *is a piecewise monotonic map. Then*

$$\limsup_{\widetilde{T} \to T \text{ in } R^0} h_{\mathrm{top}}(\widetilde{T}) = \max\left\{ h_{\mathrm{top}}(T), \max\left\{ \frac{k(x)}{n(x)} \log 2 : x \in E \text{ is periodic} \right\}\right\} .$$

Theorem 8

Let $T : [0,1] \to [0,1]$ *be a piecewise monotonic map. Suppose that* $h_{\mathrm{top}}(T) > \max\left\{ \frac{k(x)}{n(x)} \log 2 : x \in E \text{ is periodic} \right\}$ *and* μ *is the unique measure of maximal entropy for* T*. Then there exists a* $\delta > 0$ *such that every piecewise monotonic map* $\widetilde{T}$ *which is* δ*-close to* T *in the* R^0*-topology has a unique measure* $\widetilde{\mu}$ *of maximal entropy. Moreover,*

$$\lim_{\widetilde{T} \to T \text{ in } R^0} \widetilde{\mu} = \mu ,$$

where the limit is meant in the w^**-topology.*

The next result gives a condition equivalent to upper semi-continuity of the pressure for all weight functions.

Theorem 9

Let $T : [0,1] \to [0,1]$ be a piecewise monotonic map. Then the following conditions are equivalent.

(1) $\limsup_{\substack{\widetilde{T} \to T \text{ in } R^0 \\ \|\widetilde{f} - f\|_\infty \to 0}} p(\widetilde{T}, \widetilde{f}) \leq p(T, f)$ *for all continuous functions* $f : [0,1] \to \mathbb{R}$.

(2) $\lim_{\widetilde{T} \to T \text{ in } R^0} p(\widetilde{T}, f) = p(T, f)$ *for all continuous functions* $f : [0,1] \to \mathbb{R}$ *satisfying* $\liminf_{\widetilde{T} \to T \text{ in } R^0} p(\widetilde{T}, f) \geq p(T, f)$.

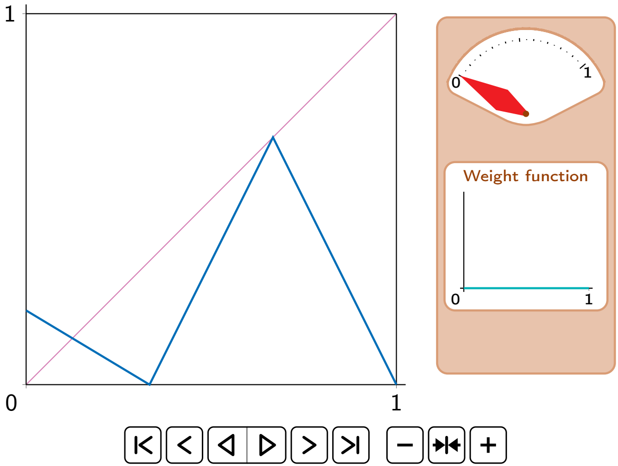
(3) *There are no periodic points in* E.

Finally we will consider some examples.

If $s \in \left[0, \frac{7}{30}\right]$ define $T_s : [0,1] \to [0,1]$ by

$$T_s x := s + \begin{cases} \frac{1}{5} - \frac{3}{5}x \,, & \text{if } x \in \left[0, \frac{1}{3}\right] , \\ 2x - \frac{2}{3} \,, & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right] , \\ 2 - 2x \,, & \text{if } x \in \left[\frac{2}{3}, 1\right] . \end{cases}$$

Here T_s is ε-close to T_0 in the R^0-topology if $s < \varepsilon$. We consider the weight function $f : [0,1] \to \mathbb{R}$ given by $f(x) := 0$ for all $x \in [0,1]$, this means we consider the entropy.


1
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Weight function
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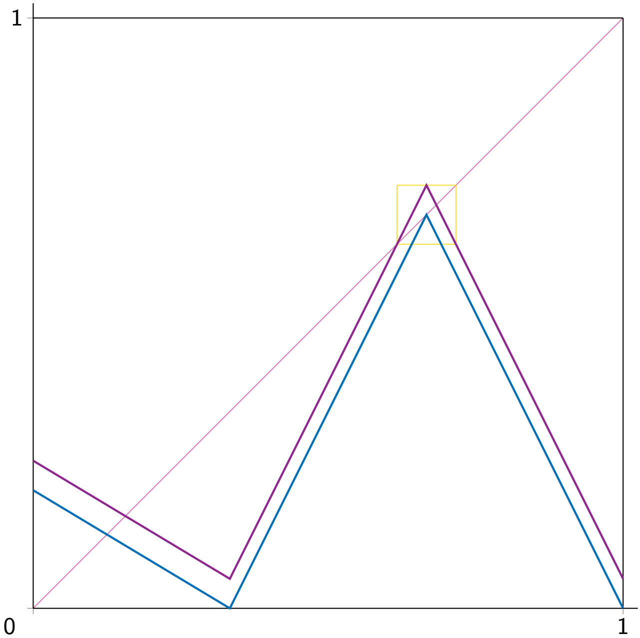
1

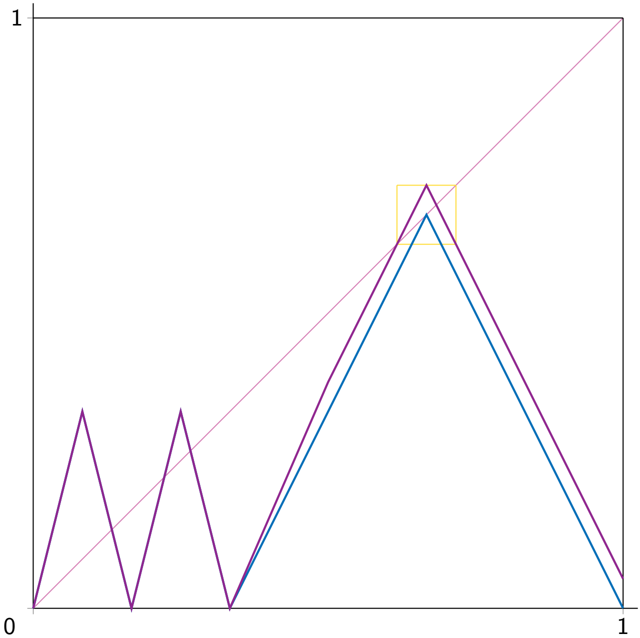


1
0
1

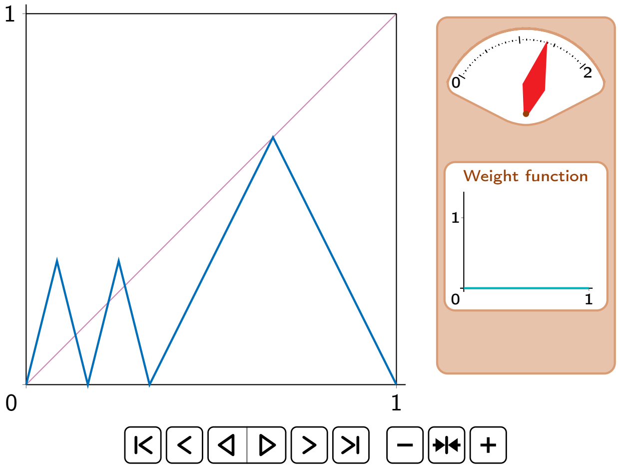
Obviously we have $h_{\text{top}}(T_0) = 0$. But for $s > 0$ we get the new invariant set $\left[\frac{2}{3} - s, \frac{2}{3} + s\right]$ and we obtain $h_{\text{top}}(T_s) = \log 2$. Define for $s \in \left[0, \frac{1}{12}\right]$ the map $T_s : [0,1] \to [0,1]$ by

$$T_s x := \begin{cases} 4x \ , & \text{if } x \in \left[0, \frac{1}{12}\right] , \\ \frac{2}{3} - 4x \ , & \text{if } x \in \left[\frac{1}{12}, \frac{1}{6}\right] , \\ 4x - \frac{2}{3} \ , & \text{if } x \in \left[\frac{1}{6}, \frac{1}{4}\right] , \\ \frac{4}{3} - 4x \ , & \text{if } x \in \left[\frac{1}{4}, \frac{1}{3}\right] , \\ (2+6s)x - \frac{2}{3} - 2s \ , & \text{if } x \in \left[\frac{1}{3}, \frac{1}{2}\right] , \\ 2x - \frac{2}{3} + s \ , & \text{if } x \in \left[\frac{1}{2}, \frac{2}{3}\right] , \\ 2 - 2x + s \ , & \text{if } x \in \left[\frac{2}{3}, 1\right] . \end{cases}$$

Also in this case T_s is ε-close to T_0 in the R^0-topology. Moreover we obtain for $s > 0$ the new invariant set $\left[\frac{2}{3} - s, \frac{2}{3} + s\right]$ from the fixed point $\frac{2}{3}$. Here we have $h_{\text{top}}(T_s) = \log 4$ for all $s \in \left[0, \frac{1}{12}\right]$.



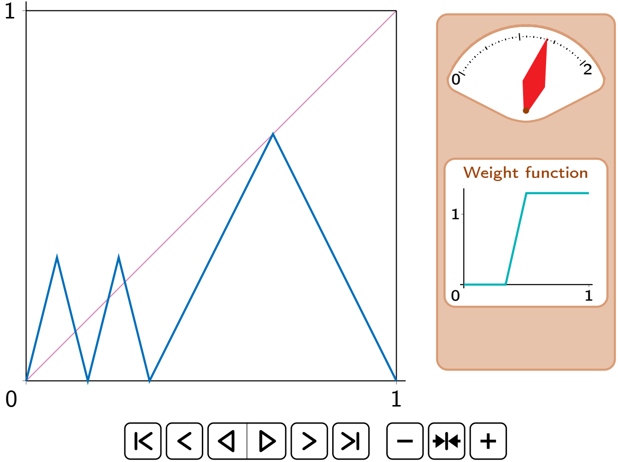
1
0
1


1
0
1
0
2
Weight function
1
0
1

We know from Theorem 9 that there exists a continuous function f such that the pressure is not upper semi-continuous at T_0. Now we give an example for such a function f. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0\,, & \text{if } x \in \left[0, \frac{1}{3}\right], \\ \frac{39}{5}x - \frac{13}{5}\,, & \text{if } x \in \left[\frac{1}{3}, \frac{1}{2}\right], \\ \frac{13}{10}\,, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

As $p(T_0, f) = h_{\text{top}}(T_0) = \log 4 > \frac{13}{10} = \sup_{x \in [0,1]} f(x)$ we get by Theorem 1 that $\liminf_{\substack{\widetilde{T} \to T \text{ in } R^0 \\ \|\widetilde{f} - f\|_\infty \to 0}} p(\widetilde{T}, \widetilde{f}) \geq p(T, f)$. For this function we get for $s > 0$ that
$p(T_s, f) = \frac{13}{10} + \log 2 > \log 4 = p(T_0, f)$.



1
0
1
0
2
Weight function
1
0
1

Thank you very much

Herzlichen Dank