Extension of mappings to non-Tychonoff spaces

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Content

Separately continuous maps

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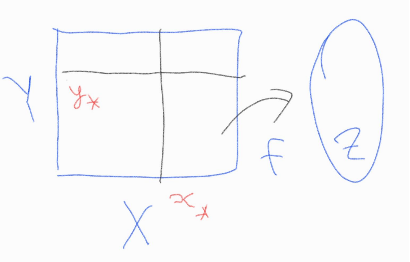
$$X \to Z, \quad x \mapsto f(x, y_*),$$
$$Y \to Z, \quad y \mapsto f(x_*, y)$$

are continuous for all $(x_*, y_*) \in X \times Y$.



Semitopological groups

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In particular, a Tychonoff countably compact semitopological group is topological. It was noted in [T2014] that the results of [KKM2001] imply that regular countably compact semitopological group is a topological group.

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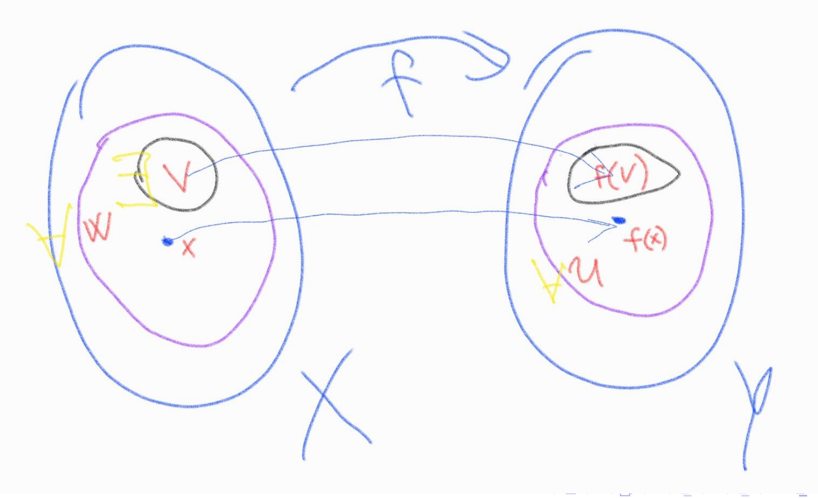
Theorem 1.2 Main Group Theorem.

Let G be a regular separable feebly compact semitopological group. Then G is a topological group.

Quasicontinuous mappings

A mapping $f : X \to Y$ is called *quasicontinuous* if for any point $x \in X$, any neighborhood $W \subset X$ of the point x and any neighborhood $U \subset Y$ of the point $f(x)$ there exists an open non-empty $V \subset W$ such that $f(V) \subset U$.



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Theorem 2.1 QuasicontinuousTheorem.

Let X, Y, Z be regular spaces, X is feebly compact, Y is a separable Baire space and $f : X \times Y \to Z$ is separately continuous. Then f is quasicontinuous.

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From Proposition 1 and Theorem 2.1 follows Main Group Theorem 1.2.

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Proof of quasicontinuity: games

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The space X is called *countably pracompact* if there exists a dense subset D of X such that D is relatively countable compact.

Let $n \in \omega$. We call a mapping $f : X \to Y$ r_n-continuous if for a sequence

$$U_0 \subset \overline{U_0} \subset U_1 \subset \overline{U_1} \subset \ldots \subset U_n$$

open subsets of Y we have $f^{-1}(U_0) \subset \operatorname{Int} f^{-1}(U_n)$.

Using the Christensen topological game, one can prove

Theorem 2.2(QuasicontinuousTheorem for pracompact X).

Let $n \in \omega$, X, Y, Z be spaces, $(x_0, y_0) \in X \times Y$, X is countably pracompact and regular at x_0, Y is a separable Baire space, Z is regular at $f(x_0, y_0)$, $f : X \times Y \to Z$,

$$X \to Z,\ x \mapsto f(x, y_*) \qquad \text{is continuos}$$
$$Y \to Z,\ y \mapsto f(x_*, y) \qquad \text{is } r_n\text{-continuos}$$

for all $(x_*, y_*) \in X \times Y$. Then f is quasicontinuous at (x_0, y_0).

Corollary 1(QuasicontinuousTheorem for compact X).

Let X, Y, Z be regular spaces, X is (countably) compact, Y is a separable Baire space and $f : X \times Y \to Z$ is separately continuous. Then f is quasicontinuous.

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For Tychonoff spaces, the corollary can be deduced from well-known theorems

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The general scheme is as follows: for a map $f : X \times Y \to Z$, we find extensions $\hat{f} : \hat{X} \times Y \to \hat{Z}$, such that $\hat{X}$ becomes "more compact" than X, and for $\hat{f}$ we can use the obtained quasicontinuity theorems.

Extension of mappings: Tychonoff spaces

Theorem 3.1 [R1994].

Let X, Y, Z be Tychonoff spaces, X is pseudocompact, Y is a separable space and $f : X \times Y \to Z$ is separately continuous. Then f can be extended to a separately continuous map $\hat{f} : \beta X \times Y \to \beta Z$.

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Proof.

It suffices to prove for the case $Z = \mathbb{R}$. Let's put

$$\Phi : X \to C_p(Y), \ x \mapsto f_{[x]} : Y \to \mathbb{R}, \ y \mapsto f(x, y).$$

Map Φ is continuos and the space $C_p(Y)$ is submetrazable (becouse Y is separable). Hence $\Phi(X)$ is (metrazable) compact and map Φ can be extended to a continuous map $\widehat{\Phi} : \beta X \to C_p(Y)$. Put $\hat{f}(x, y) = \widehat{\Phi}(x)(y)$ for $(x, y) \in \beta X \times Y$. □

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Ultrapowers and non-standard expansion

Let $p \in \beta\omega \setminus \omega$ be a free ultrafilter. Let A be a set. Consider an equivalence on A^ω: $x \sim y$ iff $\{n \in \omega : x_n = y_n\} \in p$.

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The following properties of ultrapowers are well known:

$$(A \cap B)^* = A^* \cap B^*, \quad (A \cup B)^* = A^* \cup B^*, \quad (A \times B)^* = A^* \times B^*.$$

Any mapping $f : A \to B$ extends to a mapping $f^* : A^* \to B^*$.

Let (X, τ) be a space and $U \in \tau$. On the set X^* we consider the topology whose base is formed by the sets U^* for $U \in \tau$.

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If a point $x \in X$ is regular in X, then x is regular in X^*.

The set X is relatively countably compact in X^*.

Theorem 3.2.

Let X, Y, Z be regular spaces, X is feebly compact, Y is a separable space and $f : X \times Y \to Z$ is separately continuous. Then for $f^* : X^* \times Y \to Z^*$ we have

$$X \to Z^*,\ x \mapsto f^*(x, y_*) \qquad \text{is continuos}$$
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for all $(x_*, y_*) \in X^* \times Y$.

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Question 2 [T2014].

Let G be a feebly compact regular semitopological group. Assume that G is a Fréchet space (countable tightness space, k-space). Is it true that G is a topological group?

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The answers is **YES** for Tychonoff spaces. [R1994]

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