Complements of Topologies with Short Specialization Quasiorders

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Western Kentucky University

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Intro

All spaces X are finite.
Thus, Topologies on X = Quasiorders on X.

	Order Theory	Topology
$x \lesssim y \iff$	$x \in d(y) = \downarrow y \iff$	$x \in cl\{y\}$
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$y \gtrsim x \iff$	$y \in i(x) = \uparrow x \iff$	$y \in N(x)$

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$x \approx y$ iff $x \lesssim y$ and $y \lesssim x$ is an equivalence relation.

$[x] \leq [y]$ iff $x \lesssim y$ is a partial order on $X/\approx$.

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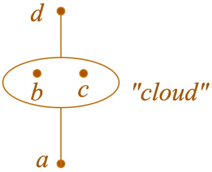
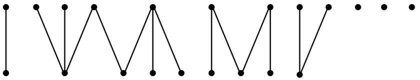


Hasse diagram for $a \lesssim b \lesssim c \lesssim b \lesssim d$

Tops with short quasiorders

Submaximal: height $h \leq 2$, partial order (no clouds)



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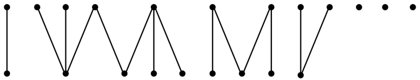
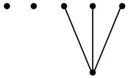
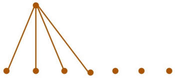


Door: $h \leq 2$, partial order, all chains have a common point



OR



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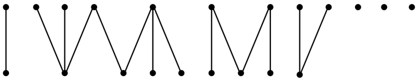

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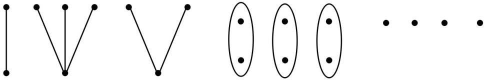
Whyburn: $|\downarrow x| \leq 2 \quad \forall x \in X$



Complements

In $TOP(X)$, $\tau = \lesssim$ and $\tau' = \lesssim'$ are complements iff

$$\tau \vee \tau' = \mathcal{P}(X) = \top \quad \text{and} \quad \tau \wedge \tau' = \{0, X\} = \bot$$

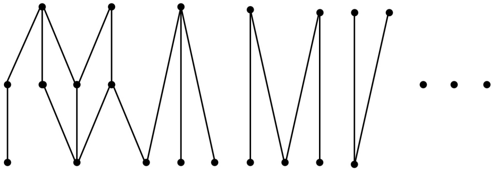
iff

$$\uparrow x \cap \uparrow' x = N(x) \cap N'(x) = \{x\} \quad \forall x \in X,$$

$$\uparrow\uparrow'\uparrow\uparrow' \cdots \uparrow\uparrow'\uparrow\uparrow' x = X \quad \forall x \in X$$

Short posets have totally ordered complements

If $\leq$ is a poset of height $h = 2$ or 3, it has a totally ordered complement.



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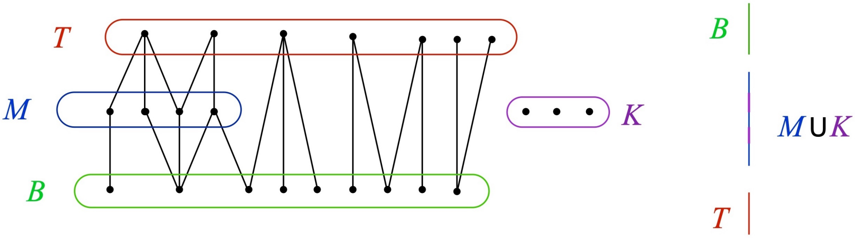
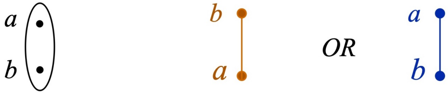


Totally ordered complements only for POsets

If quasiorder $\lesssim$ is not a p.o., then $\lesssim$ has no totally ordered complement $\leq'$



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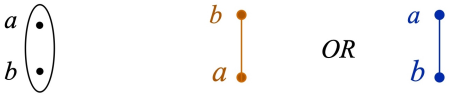

$\lesssim$ $\qquad\qquad$ $\leq'$

Case 1: $b \in \uparrow a \cap \uparrow' a$, so $\uparrow a \cap \uparrow' a \neq \{a\}$.

Case 2: $a \in \uparrow b \cap \uparrow' b$, so $\uparrow b \cap \uparrow' b \neq \{b\}$.

Finite non-discrete tops have short T_0 complements

Every non-discrete topology $\lesssim$ on a finite set X has a complement $\leq$ which is a partial order (i.e., T_0) with height $h \leq 3$.

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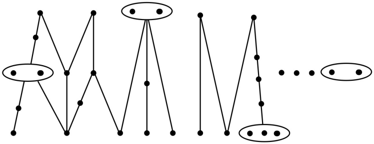

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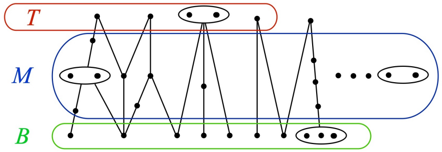

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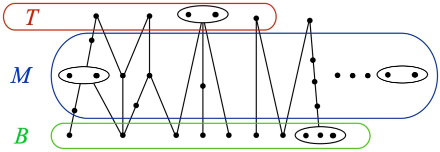

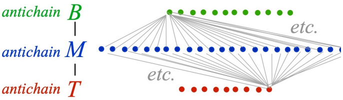

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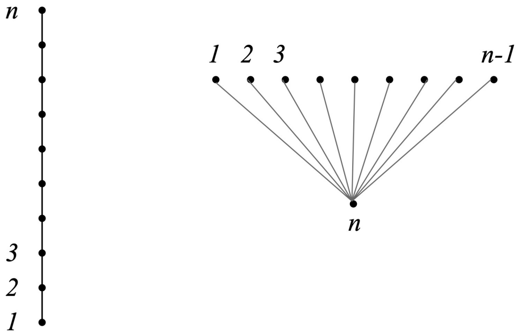
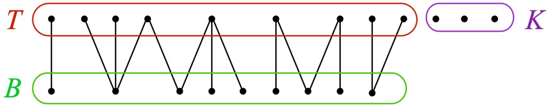


Complements of Submaximal Spaces



Theorem

Suppose $(X, \leq)$ is submaximal and not discrete. If $\leq'$ is a complement of $\leq$, then

(a) *every minimal $\leq'$-cloud contains some $t \in T$ and no points $b < t$, and*

(b) *every maximal $\leq'$-cloud contains an element of B.*

Corollary

A submaximal space $\leq'$ with isolated points has no submaximal complement $\leq$.

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Corollary

A submaximal space $\leq'$ with isolated points has no submaximal complement $\leq$.

Proof: A $\leq'$ isolated point t would be a $\leq'$-maximal and a $\leq'$-minimal cloud, so $t \in T \cap B = \emptyset$. $\square$

Sumbax comps of Submax spaces

Theorem

A submaximal space $(X, \leq)$ *has a submaximal complement iff it has no isolated points.*

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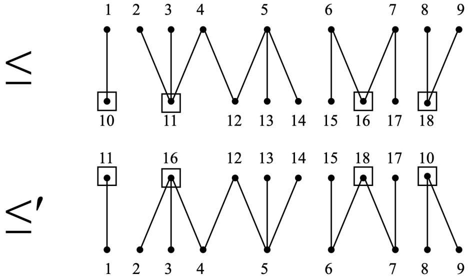
Note: Edwin Hewitt calls submaximal with no isolated points. "MI spaces".

Sumbax comps of Submax spaces

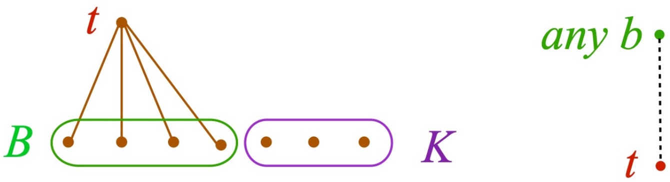
Theorem

A submaximal space $(X, \leq)$ has a submaximal complement iff it has no isolated points.

Flip $\leq$, pick one minimal point from each component, cyclically permute.



Door spaces have totally ordered complements

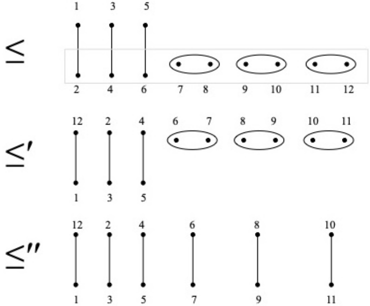


Door spaces have totally ordered complements

Indeed, if $|X| = n$ and $|B| = b$, there are $b(n-2)!$ totally ordered complements.

Whyburn Complements of Whyburn spaces

If no isolated points, no "branches" (i.e., $|B| = |T|$, $K = \emptyset$)

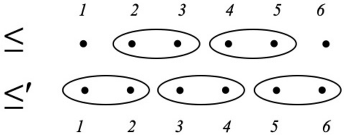


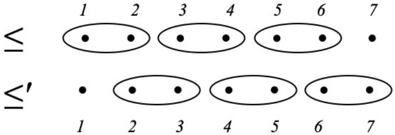
Take $\leq'$ to be $\geq$ with the boxed elements from $\leq$ cyclically permuted in $\leq'$.

$\leq''$ is a T_0 complement of $\leq$.

Whyburn Complements of Whyburn spaces

With isolated points





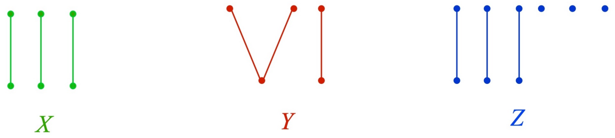
Not every Whyburn space has a Whyburn complement

Theorem

(a) *If $(X, \leq)$ is a Whyburn space with $C = \emptyset$ and $|T| \geq |B| > 1$, then $\leq$ has a Whyburn complement if and only if $|B| = |T|$ and $K = \emptyset$.*

(b) *If $(X, \leq)$ is a Whyburn space with $T = B = \emptyset$ and $4 \leq |C| \leq |K|$, then $\leq$ has no Whyburn complement.*

Example: Whyburn spaces



X has a Whyburn complement; Y and Z do not.

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