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Weakly Corson compact trees

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Background

Given a set Γ, $[0,1]^\Gamma$ is endowed with the product topology. The Σ-**product** is the dense subset

$$\Sigma(\Gamma) \coloneqq \{x \in [0,1]^\Gamma : |\operatorname{supp}(x)| \leqslant \omega\}.$$

- A compact space $\mathcal{K}$ is **Corson** if it is homeomorphic to a subset of $\Sigma(\Gamma)$, for some set Γ.
- A compact space $\mathcal{K}$ is **Valdivia** if there is a homeomorphic embedding $h\colon \mathcal{K} \to [0,1]^\Gamma$ such that $h^{-1}[\Sigma(\Gamma)]$ is dense in $\mathcal{K}$. In this case, $h^{-1}[\Sigma(\Gamma)]$ is called a Σ-**subset** of $\mathcal{K}$.
- $[0,\omega_1]$ and $[0,1]^{\omega_1}$ are Valdivia, but not Corson.
- Valdivia compacta are not closed under closed subspaces nor continuous images.

Deville, Godefroy (1993). A Valdivia compact $\mathcal{K}$ is Corson if and only if it contains no copies of $[0,\omega_1]$.

Weakly Corson compact spaces

Kalenda (2003). A compact space $\mathcal{K}$ is **weakly Corson** if it is continuous image of a countably compact subset of $\Sigma(\Gamma)$.

- Continuous image of a compact subset of $\Sigma(\Gamma)$ is Corson.
- Countably compact subset of $\Sigma(\Gamma)$ are exactly Σ-subsets of Valdivia compacta:
  - A Σ-subset of a Valdivia compact is countably compact;
  - Vice versa, if $\mathcal{X}$ is a countably compact subset of $\Sigma(\Gamma)$, $\beta\mathcal{X}$ is Valdivia and $\mathcal{X}$ is a Σ-subset of $\beta\mathcal{X}$.

  Thus, weakly Corson are related to continuous images of Valdivia compacta (also called **weakly Valdivia**).

Kalenda (2003). Let $\mathcal{K}$ be such that all closed subsets of $\mathcal{K}$ are weakly Valdivia. Is $\mathcal{K}$ weakly Corson?

- All closed subsets of $[0, \omega_1]$ are Valdivia, but $[0, \omega_1]$ is not Corson.
- $[0, \eta]$ is weakly Corson iff it is Valdivia (iff $\eta < \omega_2$).

Trees and the coarse wedge topology

- A tree is a poset $(\mathcal{T}, \leqslant)$ such that $\{s \in \mathcal{T} \colon s < t\}$ is well-ordered for every t.
- $\mathrm{ht}(t, \mathcal{T})$ is the order type of $\{s \in \mathcal{T} \colon s < t\}$.
- $\mathrm{ht}(\mathcal{T}) \coloneqq \min\{\alpha \colon \mathrm{ht}(t, \mathcal{T}) = \alpha$ for no $t \in \mathcal{T}\}$ (the **height** of $\mathcal{T}$).
- $V_t \coloneqq \{s \in \mathcal{T} \colon s \geqslant t\}$.
- The **coarse wedge topology** τ_{cw} on $\mathcal{T}$ is the one generated by V_t and their complements, where $\mathrm{ht}(t, \mathcal{T})$ is not a limit ordinal.
- $(\mathcal{T}, \tau_{\mathrm{cw}})$ is compact iff it is chain complete and it has finitely many minimal elements.
- So, we always assume that trees are chain complete and **rooted** (*i.e.*, with a unique minimal element).
- **Somaglia (2018).** There is a tree $\mathcal{T}$ with retractional skeleton, that does not contain $[0, \omega_2]$, but which is not Valdivia.

The countably coarse wedge topology

R, Somaglia (POST22). The **countably coarse wedge topology** $\tau_{\sigma\text{-cw}}$ on $\mathcal{T}$ is the one generated by V_t and their complements, where $\mathrm{ht}(t, \mathcal{T})$ does not have countable cofinality.

Theorem (R, Somaglia)

For every (chain complete and rooted) tree $\mathcal{T}$, $(\mathcal{T}, \tau_{\sigma\text{-cw}})$ is countably compact and Fréchet–Urysohn.

Given a tree $\mathcal{T}$ define $\widehat{\mathcal{T}}$ as follows: if $\mathrm{ht}(t, \mathcal{T})$ has uncountable cofinality, we add a unique point $s(t)$ between t and its predecessors $\{r \in \mathcal{T} : r < t\}$.

Theorem (R, Somaglia)

Let $\mathcal{T}$ be a tree. The following are equivalent.

(i) $(\mathcal{T}, \tau_{\mathrm{cw}})$ is weakly Corson.

(ii) $(\mathcal{T}, \tau_{\sigma\text{-cw}})$ is a Corson countably compact.

(iii) $(\widehat{\mathcal{T}}, \tau_{\mathrm{cw}})$ is Valdivia (and homeomorphic to $\beta(\mathcal{T}, \tau_{\sigma\text{-cw}})$).

Hereditarily Valdivia, not weakly Corson trees

Theorem (R, Somaglia)

If $\mathcal{T}$ is a tree of height at most $\omega_1 + 1$, every closed subset of $(\mathcal{T}, \tau_{\mathrm{cw}})$ is Valdivia.

Let $\mathcal{T}$ be the full binary tree of height $\omega_1 + 1$.

- $(\mathcal{T}, \tau_{\mathrm{cw}})$ is Valdivia.
- $\widehat{\mathcal{T}}$ is obtained from $\mathcal{T}$ adding exactly one successor to every $t \in \mathcal{T}$ with $\mathrm{ht}(t, \mathcal{T}) = \omega_1$.
- **Somaglia (2018).** $(\widehat{\mathcal{T}}, \tau_{\mathrm{cw}})$ is not Valdivia.
- By the above characterisation, $(\mathcal{T}, \tau_{\mathrm{cw}})$ is not weakly Corson.

Thank you for your attention!