Substitutions on compact alphabets

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A substitution

$\mathcal{A} = \mathbb{N}^* = \mathbb{N} \cup \{\infty\}$

$$\phi\colon \begin{cases} 0 & \mapsto & 0\ 1 \\ i & \mapsto & 0\,(i-1)\,(i+1) \\ \infty & \mapsto & 0\ \infty\ \infty \end{cases}$$

$0 \mapsto 01 \mapsto 01002 \mapsto 010020101013 \mapsto 01002010101301002010020100 2024$

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ϕ has a bi-infinite fixed point

$\cdots 01002010\infty\infty0\infty\infty010\infty\infty0\infty\infty.010020101013010020100201002024\cdots$

Question: Do frequencies of letters exist? Uniformly?
Question: What's the growth rate of $|\phi^n(i)|$? If $|\phi^n(i)| \sim \lambda^n$, what is λ?
Question: Does λ have to be algebraic?
Question: Do natural tile lengths exist?

Symbolic dynamics on compact alphabets

- $\mathcal{A}$ — **Compact Hausdorff alphabet**
- **Ex.:** S^1, $\mathbb{N}^*$, $\{0,1\}^{\mathbb{N}}$, compact manifolds, attractors, etc., ...
- $\mathcal{A}^+ = \bigsqcup_{n=1}^{\infty} \mathcal{A}^n$ — **Finite words** over $\mathcal{A}$ (product topology on $\mathcal{A}^n$)
- $\mathcal{A}^{\mathbb{Z}} = \{\cdots x_{-2}x_{-1} \cdot x_0x_1x_2 \cdots \mid x_i \in \mathcal{A}\}$ — **Full shift** on $\mathcal{A}$ with the product topology (so compact, Hausdorff)
- $S\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}\colon x_i \mapsto x_{i+1}$ — **Left shift** map, homeomorphism
- $X \subseteq \mathcal{A}^{\mathbb{Z}}$, closed, S-invariant — **Subshift**

Symbolic dynamics on compact alphabets

For $x \in \mathcal{A}^{\mathbb{Z}}$, $\quad X_x := \overline{\{S^n(x) \mid n \in \mathbb{Z}\}}$ — orbit closure

Some general properties

- Easy to show that X_x is a subshift
- $x \in X$ has dense orbit if and only if X is topologically transitive
- Every orbit in X is dense if and only if X is minimal
- $\mathcal{A}^{\mathbb{Z}}$ has a dense orbit if and only if $\mathcal{A}$ is separable
  So if X has a dense orbit, then $\mathcal{A}$ must be separable

Repetitivity

For $x \in \mathcal{A}^{\mathbb{Z}}$, $\mathcal{L}^n(x) := \{u \lhd x \, : \, |u| = n\}$ — **legal words**

Ex.: $\mathcal{A} = S^1$, $x = \cdots [\alpha^{-2}][\alpha^{-1}].[1][\alpha][\alpha^2]\cdots$, $\mathcal{L}^2(x) = \{[\alpha^i][\alpha^{i+1}]\}$

Notice that $\mathcal{L}^2(x)$ is not closed; $\overline{\mathcal{L}^2(x)} = \{[z][\alpha z] \mid z \in S^1\} \subset \mathbb{T}^2$

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$$\mathcal{L}^n(X) = \bigcup_{x \in X} \mathcal{L}^n(x)$$ — **legal words** for subshift $X \subset \mathcal{A}^{\mathbb{Z}}$

Always have $\mathcal{L}^n(X_x) = \overline{\mathcal{L}^n(x)}$ $\quad$ (need to take closure)

Definition

$x \in \mathcal{A}^{\mathbb{Z}}$ is **repetitive** if for all $n \geq 1$ and any open subset $U \subset \mathcal{L}^n(X_x)$, there exists an $N = N(n, U)$ such that for all $u \in \mathcal{L}^N(X_x)$, there is a subword of u in U.

Repetitivity

The following is well-known for finite alphabets and basically the same argument works for compact Hausdorff alphabets

Proposition

Let $x \in \mathcal{A}^{\mathbb{Z}}$. The following are equivalent:

- *x is repetitive*
- *X_x is minimal*
- *Every element of X_x is repetitive*
- *Every element of X_x has dense orbit*

Main point: this definition of repetitivity is the right one.

Substitutions

A substitution is a **continuous** function $\phi\colon \mathcal{A} \to \mathcal{A}^+$

Immediate consequences:

- Compactness of $\mathcal{A} \implies |\phi(a)|$ is bounded (normally has to be assumed)
- If $\mathcal{A}$ is connected, then ϕ has no choice, it *has* to be constant length

Ex.: $\mathcal{A} = \mathbb{N}^*, \quad \phi\colon \begin{cases} 0 & \mapsto 0\ 1 \\ i & \mapsto 0(i-1)(i+1) \\ \infty & \mapsto 0\ \infty\ \infty \end{cases}$

Ex.: $\mathcal{A} = S^1, \quad \alpha \in S^1 \qquad \phi\colon [z] \mapsto [z][\alpha z]$ (Thue–Morse-like)

Substitution subshifts

- ϕ-periodic points don't always exist — build language directly
- $u \in \mathcal{A}^n$ is **generated** by ϕ if $u \triangleleft \phi^k(a), \quad a \in \mathcal{A},\ k \geq 0$
- $\mathcal{L}(\phi) := \overline{\{u \in \mathcal{A}^+ \mid u \text{ generated}\}}$ — **Language** of ϕ
- $X_\phi := \{x \in \mathcal{A}^{\mathbb{Z}} \mid u \triangleleft x \implies u \in \mathcal{L}(\phi)\}$ — **Subshift** of ϕ

When is X_ϕ non-empty?

Theorem (Mañibo–R.–Walton '22)

X_ϕ is non-empty if and only if $\sup_{a \in \mathcal{A}, n \geq 0} |\phi^n(a)| = \infty$.

Corollary: If $\lim_{n \to \infty} |\phi^n(a)| = \infty$ for some $a \in \mathcal{A}$, then X_ϕ is non-empty

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Converse is *not* true

$\mathcal{A} = \mathbb{N}^* \times \mathbb{N}^*$

$$\phi \begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & m = 0, n = 0 \\ \begin{bmatrix} n \\ m-1 \end{bmatrix}, & m > 0 \\ \begin{bmatrix} n-1 \\ n \end{bmatrix} \begin{bmatrix} 0 \\ n \end{bmatrix}, & m = 0, n > 0 \end{cases}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$X_\phi = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}^\infty \}$

Primitivity

[Durand–Ormes–Petite '18, Frank–Sadun '14, Queffelec '87]

ϕ is **primitive** if for every open $U \subset \mathcal{A}$, there is $k = k(U)$ such that for all $a \in \mathcal{A}$, some letter of $\phi^k(a)$ is in U

Ex.: $\mathcal{A} = S^1, \quad \phi \colon [\theta] \to [\theta][\alpha\theta]$ for α irrational

Kronecker's theorem $\implies$ primitive

Ex.: $\mathcal{A} = \mathbb{N}^*, \quad \phi \colon n \to 0 \; n+1$

Given k, $\phi^k(n)$ hits the open interval $[k, \infty]$, so primitive.

Ex.: $\mathcal{A} = \mathbb{N}^*, \quad \phi \colon n \to n-1 \; n \; n+1$

Not primitive because you have to take different powers to hit 0 depending on n (also $\infty \mapsto \infty \; \infty \; \infty$)

Primitivity

Theorem (Durand–Ormes–Petite '18)

If ϕ is primitive, then X_ϕ is minimal.

Hence every $x \in X_\phi$ has dense orbit and is repetitive

Question: Primitive $\implies$ X_ϕ uniquely ergodic?

Primitivity

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Question: Primitive $\implies$ X_ϕ uniquely ergodic?

Answer: No! Counterexample given by DOP.

We need stronger conditions.

Perron–Frobenius?

What about Perron–Frobenius theory, expansion factors, frequencies, natural tile-lengths, unique ergodicity, etc...?

Have to consider the operator

$$T\colon C(\mathcal{A}, \mathbb{R}) \to C(\mathcal{A}, \mathbb{R})$$

$$Tf(a) = \sum_{i=1}^{|\phi(a)|} f(\phi(a)_i)$$

T is a **positive** operator because $f \geq 0 \implies Tf \geq 0$

Note: T is the **transpose** of the usual substitution matrix when $\mathcal{A}$ is finite

Abelianisation

In general, exists no power p such that for all $f \geq 0$, $T^p f > 0$.

Too much to ask for! Perron–Frobenius theory **fails** in infinite dimensions

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An operator T is **strongly positive** if for all $0 \neq f \geq 0$, there exists $k = k(f)$ such that $T^k f > 0$

Proposition (Mañibo–R.–Walton '22)

ϕ is primitive if and only if T is strongly positive.

Allows us to use the theory of strongly positive operators on Banach lattices (rich, beautiful theory from 60s & 70s — Abdelaziz, Karlin, Schaefer,...)

Length functions

A continuous $L\colon \mathcal{A} \to \mathbb{R}$ is a **length function** if for all $a \in \mathcal{A}$,

$$\sum_{i=1}^{|\phi(a)|} L(\phi(a)_i) = \lambda L(a), \qquad \lambda > 1$$

Gives consistent geometric inflation rule
Inflation factor λ

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Inflation factor λ

$$L \text{ is a length function} \iff L \text{ is an eigenfunction of } T \ (\lambda > 1)$$

Really, we want L to be strictly positive

Invariant measures

Frank and Sadun's 'fusion' framework ($+$ some technicalities) shows that invariant measures for X_ϕ correspond to eigenmeasures of T'
(need the existence of a length function $L > 0$ with $\lambda > 1$)

$$T' \text{ has a unique positive eigenmeasure} \iff X_\phi \text{ is uniquely ergodic}$$

Main results

T is called **quasi-compact** if $\| T^n - C \| < \rho^n$ for compact C and $n \geq 1$

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Theorem (Mañibo–R.–Walton '22)

Suppose ϕ is primitive and T is quasi-compact. Then:

- *ρ is isolated in the spectrum of T*
- *ρ is a pole of the resolvent*
- $|\phi^n(a)| \sim \rho^n$
- *ϕ has a continuous length function L with $\lambda = \rho > 1$*
- $L > 0$
- *L is unique (up to rescaling)*
- *L is the only strictly positive eigenfunction*
- *T' admits a unique positive eigenmeasure (up to rescaling) — so X_ϕ is uniquely ergodic*

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So it's important to find conditions which ensure T is quasi-compact!

Main results

Theorem (Mañibo–R.–Walton '22)

Suppose there is a finite set of isolated points $P \subset \mathcal{A}$ and $k \geq 0$ such that for all $a \in \mathcal{A}$

$$\#\{b \triangleleft \phi^k(a) \mid b \notin P\} < \rho^k.$$

Then T is quasi-compact.

Condition says if we look at large enough ϕ^k, few land outside of P.

Making P larger or k greater makes it easier to satisfy the bounds.

It would be good if we could replace P isolated & finite with a weaker condition.

Actually, we can drop the isolated condition, but we pay with a factor of 2 on the LHS

Applying to an example

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Take $P = \{0\}$.

Easy to see that $\rho^2 \geq 5$ because $\min_{a \in \mathcal{A}} |\phi^2(a)| = 5$.

We have $\#\{b \triangleleft \phi(a) \mid b \neq 0\} \leq 2 < \sqrt{5} \leq \rho$, so condition holds.

Hence X_ϕ is uniquely ergodic!

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Hence X_ϕ is uniquely ergodic!

In fact, $\rho = \frac{5}{2}$, and we can find (with some difficulty) the frequencies and natural tile lengths of the letters.

$$\mu[i] = 2^{-(i+1)}, \qquad L(i) = 2 - 2^{-i}$$

Quasicompactness isn't enough!

$\mathcal{A} = S^1$

$$[z] \mapsto [z][\alpha z]$$

Then T is not quasi-compact!

T is **strongly power convergent** though, and X_ϕ is uniquely ergodic

Can show that if the columns of a primitive, constant length substitution generate a uniformly equicontinuous family then T is strongly power convergent, which is also enough for unique ergodicity
(similar to result of [Queffelec '87])

Question: What condition P on T gives
"primitivity $+ P \implies$ unique ergodicity"?

Thank you!

unless I have extra time?

Spectral gap

[Frank–Sadun '14]
Without continuity, one can get ρ not isolated — this causes lots of problems!

Question: Do our conditions stop this from happening?

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Theorem (Mañibo–R.–Walton '22)

If ϕ is primitive and T is quasi-compact, then T has a spectral gap. That is, there is $r < \rho$ such that $\sigma \setminus \{\rho\} \subset B_r(0)$.

Follows from classic result that T strongly positive and quasi-compact $\implies$ spectral gap.

Discrepancy estimates

If we have a spectral gap, then for all $f \in C(\mathcal{A})$, $a \in \mathcal{A}$, $n \geq 1$

$$\mathrm{Err}(f,a,n) := |\overbrace{\rho^n L(a) \cdot \mu(f)}^{\text{expected value}} - \overbrace{T^n f(a)}^{\text{actual value}}| = O(r^n)$$

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This is a key property used in the finite alphabet setting as it bounds '**discrepancy estimates**'. Will be useful in the future.

Means we can look at questions like weak-mixing, topological mixing, when a tiling is bounded distance to a lattice, cut-and-project methods, what it means to be a 'Pisot' substitution, (i.e., when $r < 1$), etc., ...

Thank you!