Asymptotic invariants of measure preserving vector fields

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Motivation

Euler's equations for the velocity v_t of a perfect fluid :

$$\begin{cases} \nabla \cdot v_t &= 0 \\ \frac{\partial v_t}{\partial t} + (v_t \cdot \nabla) v_t + \nabla p &= \vec{0} \end{cases}$$

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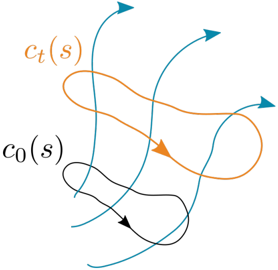

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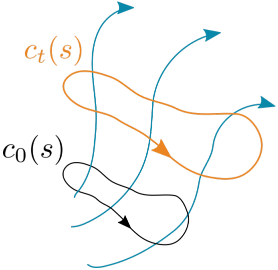

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Given X preserving μ on $\mathbb{S}^3$,
can we construct invariants by μ-preserving diffeomorphism ?

Helicity and linking number

Suppose μ is a volume.

Proposition

$Hel(X, \mu) = \int_{\mathbb{S}^3} \alpha_X \wedge d\alpha_X$ is the helicity of X, where $d\alpha_X = i_X\mu$.

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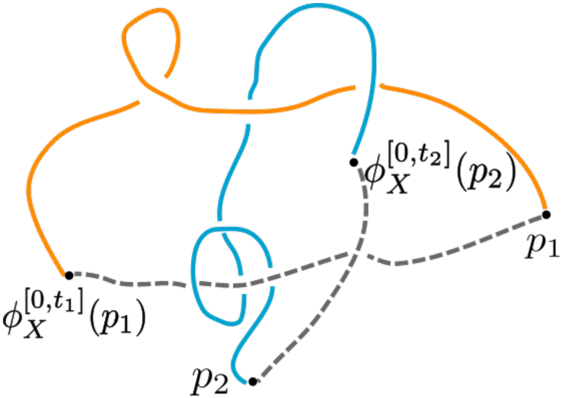

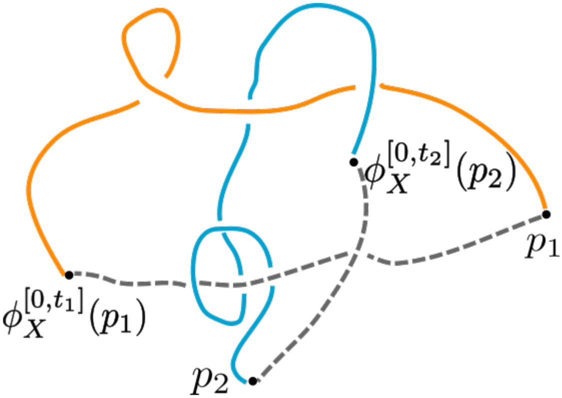

Theorem (Arnold-Vogel)

Suppose μ is a volume. Then μ-almost everywhere we have :

$$Lk_X(p_1, p_2) := \lim_{t_1, t_2 \to \infty} \frac{1}{t_1 t_2} Link(k_X(p_1, t_1), k_X(p_2, t_2))$$

and the average of the quantity $Lk_X(p_1, p_2)$ is equal to $\frac{1}{\mu(\mathbb{S}^3)} Hel(X, \mu)$.

- Take a link (or knot) invariant I ;
- Form the knots $k_X(p_i, T_i)$;
- If for almost $(p_1, ..p_n)$, $I(k_X(p_1, T_1), ..., k_X(p_n, T_n))$ is asymptotically $I_\infty(p_1, ...p_n) \times T_1^{m_1} ... T_n^{m_n}$ and if the function $(p_1, ..., p_n) \mapsto I_\infty(p_1, ...p_n)$ is integrable with respect to μ, then its integral is an invariant !

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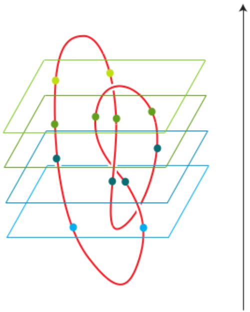
Theorem (Kudryavtseva'15, Encisco-Peralta-Torres'16)

If X is ergodic for μ, every regular integral invariant is a $\mathcal{C}^1$ function of helicity.

Even if we ask for less regular invariants, some turn out to be function of helicity when X is ergodic for μ...

The trunkenness of vector fields [Dehornoy-Rechtman'17]



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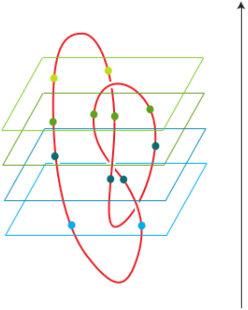

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Definition

The trunk of a knot k is given by

$$Trunk(k) = \min_{h \text{ height fct}} \max_{t \in]0,1[} \sharp\{h^{-1}(t) \cap k\}\,.$$

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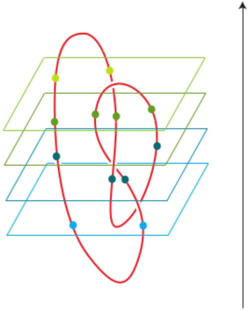

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Definition (Generalisation)

Let X be a vector field on $\mathbb{S}^3$ preserving a probability measure μ. The trunkenness of (X, μ) is given by

$$Tks(X, \mu) = \inf_{h \text{ height function}} \max_{t \in [0,1]} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mu\left(\phi_X^{[0,\epsilon]}\left(h^{-1}(t)\right)\right) .$$

The trunkenness of vector fields [Dehornoy-Rechtman'17]

Theorem (Dehornoy-Rechtman'17)

Invariance : *The trunkenness is invariant by μ-preserving homeomorphisms.*

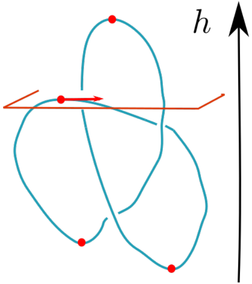
Continuity : *There is a topology for the vector field and the measure which makes the trunkenness continuous in some sense.*

Asymptotic : *For μ-almost every $p \in \mathbb{S}^3$, the limit*

$$\lim_{t \to \infty} \frac{1}{t} Trunk(k_X(p, t))$$

exists and is equal in average to $Tks(X, \mu)$.

Bridge number of vector fields [R. '21]



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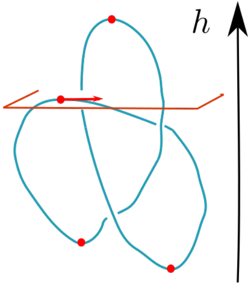

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The bridge number of the knot k is given by

$$Bridge(k) = \min_{h \text{ height fct}} \frac{1}{2} \sharp \{ Local\ extrema\ of\ h_{|k} \} .$$

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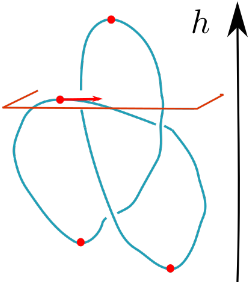

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The bridge number of (X, μ) is given by

$$b(X,\mu) = \inf_{h \text{ height function}} \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mu \left(\phi_X^{[0,\epsilon]} \left(\cup_{t=0}^{1} T_X \left(h^{-1}(t) \right) \right) \right) .$$

Bridge number of vector fields [R. '21]

Theorem (R.'21)

Invariance : *The bridge number of vector fields is invariant by μ-preserving $\mathcal{C}^1$-diffeomorphisms.*

Continuity : *Let (X_n, μ_n) a sequence of measure-preserving vector fields such that $(X_n)_{n\in\mathbb{N}}$ tends to X in the $\mathcal{C}^0$ topology and $(\mu_n)_{n\in\mathbb{N}}$ converges to μ in the weak* sense. Then*

$$\lim_{n\to\infty} b(X_n, \mu_n) = b(X, \mu) .$$

Asymptotic : *For μ-almost every $p \in \mathbb{S}^3$, the limit*

$$\lim_{t\to\infty} \frac{1}{t} Bridge(k_X(p, t))$$

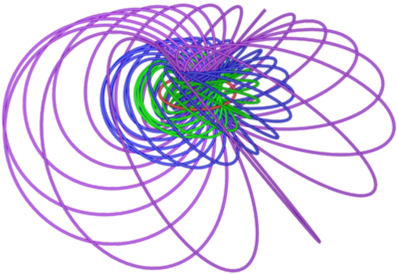
exists and is equal in average to $b(X, \mu)$.

Example

Seifert flow on $\mathbb{S}^3$ of parameters $(\alpha, \beta) \in (\mathbb{R}_+^*)^2$.

$$\phi_{\alpha,\beta}^t(z_1, z_2) = \left(e^{i\alpha t} z_1, e^{i\beta t} z_2\right)$$

- Torii $\left|\frac{z_1}{z_2}\right| = c$ are invariant ;



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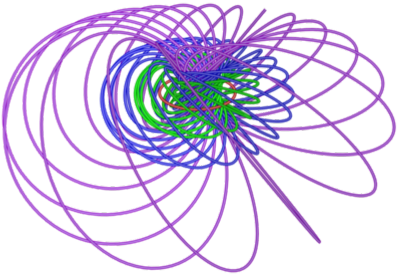

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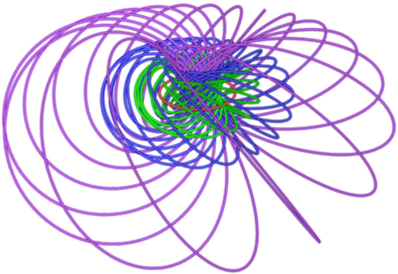
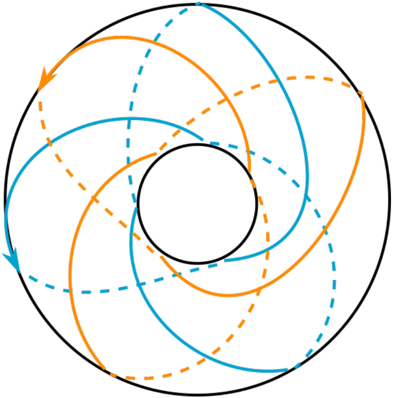
- If α/β is rational, orbits are torus knots ;
- Preserves the volume (Haar measure) ; non-ergodic but can be $\mathcal{C}^1$-perturbated into an ergodic vector field.



Independance

- $\mathrm{Hel}(X, \mathit{Vol}) = \alpha\beta$
- $b(X, \mathit{Vol}) = \min\{\alpha, \beta\}$
- $\mathrm{Tks}(X, \mathit{Vol}) = 2\min\{\alpha, \beta\}$



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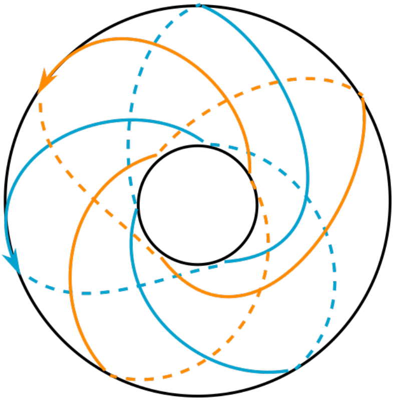
Independance of the trunkenness and the bridge number ?



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Independance of the trunkenness and the bridge number ?

- $Bridge(k_1 \sharp k_2) = Bridge(k_1) + Bridge(k_2) - 1.$
- $Trunk(k_1 \sharp k_2) = \max\{Trunk(k_1), Trunk(k_2)\}.$

Further questions

Theorem (Dehornoy-Rechtman'17)

Let X be a non-singular vector field on $\mathbb{S}^3$ preserving the measure μ and h a height function such that

$$Tks(X, \mu) = \max_{t \in [0,1]} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mu \left(\phi_X^{[0,\epsilon]} \left(h^{-1}(t) \right) \right) .$$

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Then X has an unknotted periodic orbit.

- Could the difference $2b - Tks$ indicate the existence of composite knots among the orbits ?
- Are the trunkenness or the bridge number related to energy ?
- Generalisation of these invariants to foliations of higher dimension in spaces of higher dimension ?

Thank you for listening !