Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Mass Transport on Aperiodic Tilings

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Cohomology solves transport
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Outline

1. Framing the question

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Outline

1. Framing the question

2. Some puzzles

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Outline

1. Framing the question

2. Some puzzles

3. Pattern-Equivariant Cohomology

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Outline

1. Framing the question

2. Some puzzles

3. Pattern-Equivariant Cohomology

4. Cohomology solves transport

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Outline

1. Framing the question

2. Some puzzles

3. Pattern-Equivariant Cohomology

4. Cohomology solves transport

5. Answers to puzzles

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Outline

1. Framing the question

2. Some puzzles

3. Pattern-Equivariant Cohomology

4. Cohomology solves transport

5. Answers to puzzles

6. Conjectures, partial results and open problems

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Table of Contents

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

The question

Given a (repetitive, uniquely ergodic, aperiodic) tiling T and two locally-determined mass distributions f_1, f_2 on T:

- When is it possible to do a bounded transport from f_1 to f_2?

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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Given a (repetitive, uniquely ergodic, aperiodic) tiling T and two locally-determined mass distributions f_1, f_2 on T:

- When is it possible to do a bounded transport from f_1 to f_2?
- When can that bounded transport be done in a strongly pattern-equivariant way?

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- When can it be done in a weakly pattern-equivariant way?

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- When is it possible to do a bounded transport from f_1 to f_2?
- When can that bounded transport be done in a strongly pattern-equivariant way?
- When can it be done in a weakly pattern-equivariant way?
- What does this have to do with cohomology?

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Defining our terms

- A *mass distribution f* on *T* is an assignment of a non-negative number to each tile, representing the total amount of mass in that tile.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Defining our terms

- A *mass distribution* f on T is an assignment of a non-negative number to each tile, representing the total amount of mass in that tile.
- The distribution f is *strongly pattern equivariant* (sPE) if $\exists R > 0$ such that for each tile t, $f(t)$ is determined exactly by the pattern of T in an R-neighborhood of t.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- The distribution f is *weakly pattern equivariant* (wPE) if it is the uniform limit of sPE distributions.

- A *bounded transport* from f_1 to f_2 is a set of rules for moving mass around (e.g. send 10 kg from tile t_1 to tile t_2, 3kg from t_2 to t_3, and 5 kg from t_3 to t_1) such that
  - The motion changes f_1 to f_2, and
  - No mass is moved by more than a fixed distance D

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems
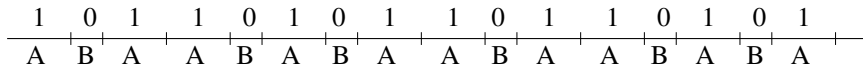
Pattern equivariant transport

- The transport is sPE if there is a radius R such that the transport from each t_i to t_j is determined exactly by the pattern of T in an R-neighborhood of t_i.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Pattern equivariant transport

- The transport is sPE if there is a radius R such that the transport from each t_i to t_j is determined exactly by the pattern of T in an R-neighborhood of t_i.

- wPE is the uniform limit of sPE. For any ϵ there exists R_ϵ such that the transport from t_i to t_j is determined within ϵ by the R_ϵ neighborhood of t_i.

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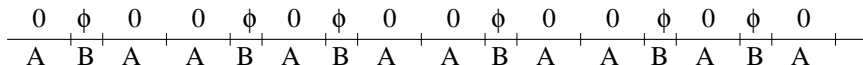Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Table of Contents

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

## Two Fibonacci mass distributions

Put 1kg on each *A* tile:

1		0		1		1		0		1		0		1		1		0		1		1		0		1		0		1

A B A A B A B A A B A A B A B A

Put $\phi = (1 + \sqrt{5})/2$ kg on each *B* tile.

0		ϕ		0		0		ϕ		0		ϕ		0		0		ϕ		0		0		ϕ		0		ϕ		0

A B A A B A B A A B A A B A B A

Same overall density. Is there bounded/wPE/sPE transport?

Lorenzo Sadun   Mass Transport on Aperiodic Tilings

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

A chair puzzle

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems
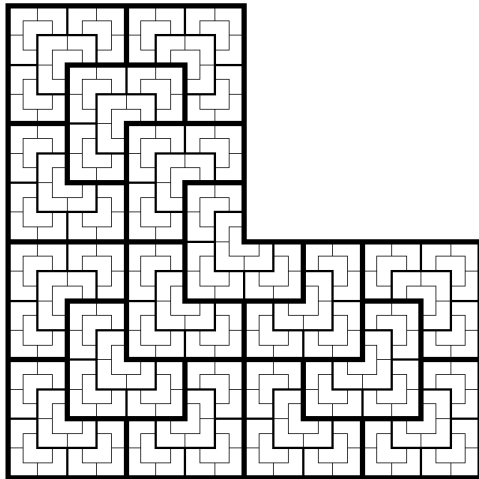
f_1 places 2 kg on each NE chair

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems
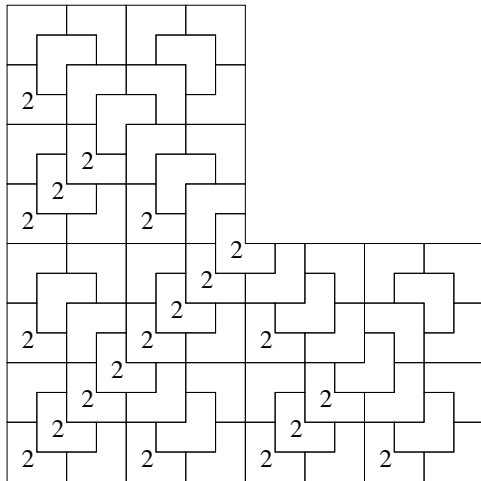
f_2 places 1 kg on each NE and SW chair

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

f_3 places 1 kg on each NW and SE chair

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Three different mass distributions

- All three distributions have overall density 0.5 kg/tile.
- Which are related by bounded/wPE/sPE transport?

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Table of Contents

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Pattern-equivariant cochains

- A tiling T gives a decomposition of $\mathbb{R}^n$ into vertices, edges, 2-cells, 3-cells, etc. Tiles are n-cells. Orient the cells arbitrarily.

- A (real-valued) k-cochain assigns a real number to each oriented k-cell. A mass distribution is just an n-cochain.

- k-cochains can be bounded/wPE/sPE.

- Coboundaries: If α is a k-cochain, and c is a $(k+1)$-cell, then $(\delta\alpha)(c) := \alpha(\partial c)$.

- If α is bounded/wPE/sPE, so is $\delta\alpha$, albeit with bigger radius.

- Let Ω_w^k and Ω_s^k denote the weakly and strongly PE k-cochains on T.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

(Strong) PE cohomology

A strongly PE cochain α is said to be

- **Closed** is $\delta\alpha = 0$,

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- **Closed** is $\delta\alpha = 0$,
- **Exact** if $\alpha = \delta\beta$ for some sPE cochain β,

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- $H_{PE}^k = \dfrac{\text{Closed } k\text{-cochains}}{\text{Exact } k\text{-cochains}}$

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- **Weakly exact** if $\alpha = \delta\gamma$ for some wPE cochain γ.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- $H_{PE}^k = \dfrac{\text{Closed } k\text{-cochains}}{\text{Exact } k\text{-cochains}}$
- **Weakly exact** if $\alpha = \delta\gamma$ for some wPE cochain γ.
- A cohomology class is **asymptotically negligible (AN)** if it can be represented by a weakly exact cochain.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

A topological invariant

Theorem (Kellendonk-Putnam, S)

If T is a repetitive tiling, then H_{PE}^k is canonically isomorphic to the k-th real-valued Čech cohomology $\check{H}^k(\Omega_T)$, where Ω_T is the continuous hull of T. In particular, all tilings in Ω_T have the same PE cohomology.

A lot is known about cohomology of tiling spaces. If we can reduce questions of transport to questions of cohomology, we win.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Integration and well-balanced cochains

- k-cochains are made to be integrated on k-chains. If α is an n-cochain and X is a union of tiles, then

$$\alpha(X) = \sum_{t \in X} \alpha(t).$$

- α is *well-balanced* if $\exists C < \infty$ such that, for any finite union X of tiles,

$$|\alpha(X)| \leq C|\partial X|.$$

- Adding $\delta\beta$ to α does not change well-balancedness, since $|\delta\beta(X)| = |\beta(\partial X)| \leq C|\partial X|$. Well-balancedness is a property of cohomology *classes*, not just cochains.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Table of Contents

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Cohomological answers to transport questions

If f_1 and f_2 are mass distributions on T, then f_1 and f_2 are closed and define cohomology classes $[f_1]$ and $[f_2]$. Then

- Theorem: There is a bounded transport from f_1 to f_2 if and only if $[f_1 - f_2]$ is well-balanced.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- Theorem: There is a bounded transport from f_1 to f_2 if and only if $[f_1 - f_2]$ is well-balanced.
- There is a wPE transport from f_1 to f_2 if and only if $f_1 - f_2$ is weakly exact.
- There is a sPE transport from f_1 to f_2 if and only if $f_1 - f_2$ is exact, i.e. if and only if $[f_1] = [f_2]$.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Laczkovich and Hall

- For fixed D, let X_{+D} be the union of all tiles within distance D of the tiles of X.

- If there is a bounded transport from f_1 to f_2 moving mass a maximum distance D, then for any X, $f_1(X_{+D}) \geq f_2(X)$ and $f_2(X_{+D}) \geq f_1(X)$.

- Using the Hall Marriage Theorem, Laczkovich proved that this condition is sufficient for the existence of bounded transport.

- We need to show that the Laczkovich condition is equivalent to well-boundedness.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Trains and border crossings

Imagine supply depots in each tile. WLOG assume that f_1 and f_2 are strictly positive and bounded below by ϵ. Ship goods by train to nearby tiles. Customs agents keep track of how much mass crosses each border.

- If $f_1 - f_2$ is well-balanced, and if $C < N\epsilon$ for some integer N, and if the tiles have a maximum of M faces, then we can accomplish the mass transport from f_1 to f_2 by interpolating linearly in NM steps. At each step the Lackovich condition is met with D being the diameter of the largest tile.

- If f_1 and f_2 satisfy the Laczkovich condition, then there is a bounded transport. For each edge e let $\beta(e)$ be the net amount of mass crossing the edge. Then $\delta\beta = f_1 - f_2$. Since β is bounded, $|(f_1 - f_2)(X)| = |\beta(\partial X)| \leq C|\partial X|$.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

PE border crossings

- If the transport is (weakly or strongly) PE, then β is (weakly or strongly) PE, and $f_1 - f_2$ is (weakly or strongly) exact.

- Conversely, if $f_1 - f_2 = \delta\beta$ for some PE cochain β, then β gives the instructions for mass transport. If there's not enough mass in some tiles to transport across an edge in one step, use multiple steps.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Table of Contents

1. Framing the question

2. Some puzzles

3. Pattern-Equivariant Cohomology

4. Cohomology solves transport

5. Answers to puzzles

6. Conjectures, partial results and open problems

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Fibonacci puzzle

versus

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Fibonacci answer

- For the Fibonacci tiling, $\check{H}^1(\Omega_T) = \mathbb{Z}^2$, so $\check{H}^1(\Omega_T, \mathbb{R}) = \mathbb{R}^2$.
- $H^1_{AN} = \mathbb{R}$. The difference of two sPE mass distributions is weakly exact if and only if they have the same density.
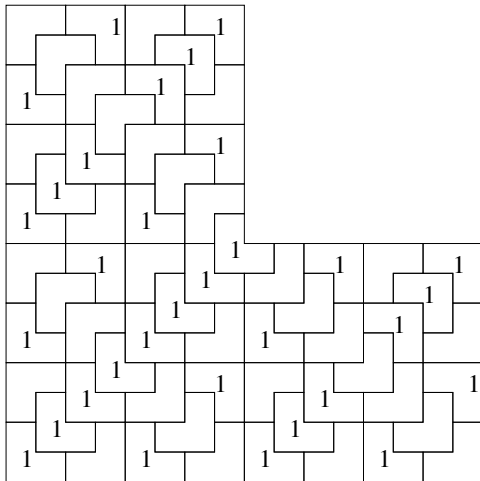- So there is a wPE transport, and hence a bounded transport. But is there a sPE transport?

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- $H^1_{AN} = \mathbb{R}$. The difference of two sPE mass distributions is weakly exact if and only if they have the same density.

- So there is a wPE transport, and hence a bounded transport. But is there a sPE transport?

- No! If $f_1 - f_2 = \delta\beta$ and β is sPE, then β takes on the exact same value repeatedly, say at positions a and b. But then $(f_1 - f_2)[a, b] = 0$. But $(f_1 - f_2)$ is a positive multiple of 1 minus a positive multiple of ϕ, and cannot be zero.

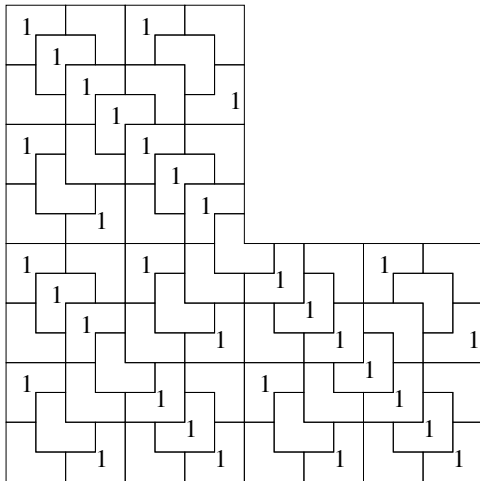Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

f_1: 2 kg on the NE chairs

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems
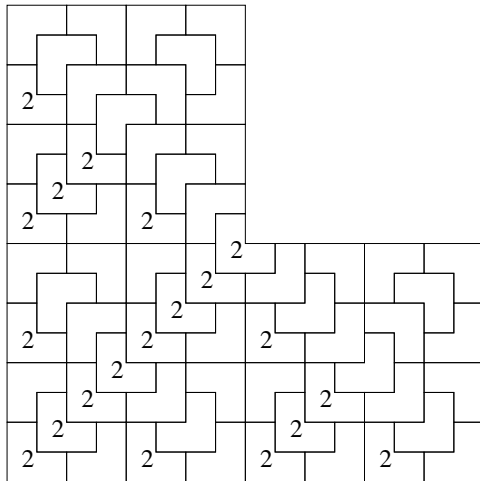
f_2: 1 kg on the NE and SW chairs

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

f_3: 1 kg on the NW and SE chairs

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Chair answers

- For the chair tiling, H^2_{AN} is trivial and $H^2(\Omega_T, \mathbb{R}) = \mathbb{R}^3$.
- One generator counts all tiles equally. Not well-balanced.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Chair answers

- For the chair tiling, H^2_{AN} is trivial and $H^2(\Omega_T, \mathbb{R}) = \mathbb{R}^3$.
- One generator counts all tiles equally. Not well-balanced.
- NE + SW - SE - NW is cohomologically trivial. Every 1-supertile has exactly two (NE or SW) and two (NW + SE). To get sPE transport, just move mass within each 1-supertile.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
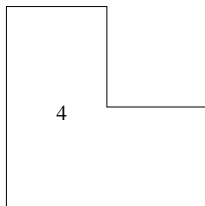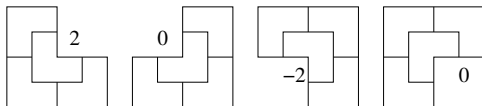Conjectures, partial results and open problems

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- One generator counts NE minus SW. This is $f_1 - f_2$. Not weakly exact, so there is no wPE transport.
- (Last generator counts NW minus SE.)

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- One generator counts NE minus SW. This is $f_1 - f_2$. Not weakly exact, so there is no wPE transport.
- (Last generator counts NW minus SE.)
- Remaining question: Is $f_1 - f_2$ well-balanced?

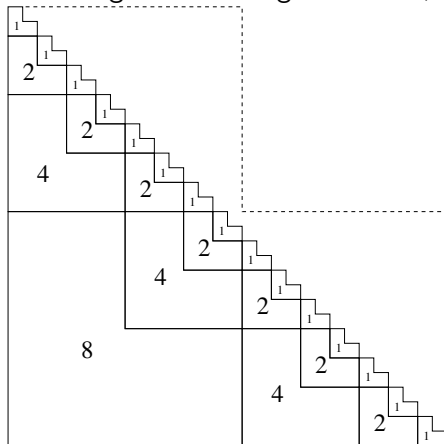Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Scaling properties

Under substitution, $f_1 - f_2$ doubles at each stage:

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

N log N

On triangle of side length $N = 2^m$, $f_1 - f_2$ goes as $m2^m$.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Table of Contents

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Equivalence?

- sPE transport $\implies$ wPE transport $\implies$ bounded transport.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Equivalence?

- sPE transport $\implies$ wPE transport $\implies$ bounded transport.
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Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Equivalence?

- sPE transport $\implies$ wPE transport $\implies$ bounded transport.
- $[\alpha] = 0 \implies \alpha$ is weakly exact $\implies \alpha$ is well-balanced.
- But what about the converses?

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

sPE vs. wPE

Existence of wPE transport does not imply existence of sPE transport.

AN classes exist. Fibonacci puzzle has a wPE solution but not a sPE solution.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

wPE vs. bounded transport

Bounded transport is not necessarily wPE.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

wPE vs. bounded transport

Bounded transport is not necessarily wPE.

Take the standard square lattice on $\mathbb{R}^2$. Let $f_1 = f_2 =$ constant.
(E.g. 1kg on each tile). Move mass from $(0,0)$ to $(0,1)$ to $(1,1)$
to $(1,0)$ to $(0,0)$.

- Initial and final mass distributions are sPE. (Constant!)
- Transport process is not wPE.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

wPE vs. bounded transport

Bounded transport is not necessarily wPE.

Take the standard square lattice on $\mathbb{R}^2$. Let $f_1 = f_2 =$ constant.
(E.g. 1kg on each tile). Move mass from $(0,0)$ to $(0,1)$ to $(1,1)$
to $(1,0)$ to $(0,0)$.

- Initial and final mass distributions are sPE. (Constant!)
- Transport process is not wPE.
- Problem is gauge freedom. Need a rule to fix the curl around
  each vertex.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Conjecture 1

Let f_1 and f_2 be sPE mass distributions for a repetitive and uniquely ergodic tiling T. If there exists a bounded transport from f_1 to f_2, then there exists a (possibly different) wPE transport from f_1 to f_2.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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Cohomological restatement:

If α is a well-balanced sPE n-cochain on Ω_T, then α is asymptotically negligible.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Expanding the question

Generalize definitions of well-bounded and asymptotically negligible to cohomology classes of all degrees:

- A closed k-cochain is *well-balanced if its integral over any k-chain X (union of k-faces of tiles in T) is bounded by the boundary of X.*
- A class in H^k is well-balanced if it is represented by a well-balanced cochain.
- A class is H^k is asymptotically negligible if it is represented by a weakly exact cochain.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- A class is H^k is asymptotically negligible if it is represented by a weakly exact cochain.

More general conjecture: Every well-balanced cohomology class of arbitrary degree is asymptotically negligible.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Status of the conjecture

- True when $k = 1$, independent of n. (Gottschalk-Hedlund theorem. Tiling version by Kellendonk-S) A closed sPE 1-cochain is weakly exact if and only if its integral is bounded.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- True when T is a codimension-1 canonical projection tiling ("stepped plane"). H^k is generated by H^1 and H^1 is spanned by dx^1, dx^2, ..., dx^n and $d\beta$ where β is wPE.
  $d\beta \wedge \gamma = d(\beta \wedge \gamma)$.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- Partial results when T is a self-similar (substitution) tiling.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Results for substitution tilings

- T is a substitution tiling with stretching factor λ.
  Substitution map $\sigma : \Omega_T \to \Omega_T$ induces map
  $\sigma^* : H^k(\Omega_T, \mathbb{R}) \to H^k(\Omega_T, \mathbb{R})$. Decompose $H^k(\Omega_T, \mathbb{R})$ into
  (generalized) eigenspaces of σ^*.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- If eigenvalue μ has $|\mu| < \lambda^{k-1}$, then class $[\alpha]$ is AN, hence
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Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- If $|\mu| > \lambda^{k-1}$, or if $|\mu| = \lambda^{k-1}$ and $[\alpha]$ isn't an eigenvector, then $[\alpha]$ is not WB and is not AN.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Open question

But in the borderline case, could $[\alpha]$ still be WB?

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Open question

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Lots of interesting examples, like chair and Penrose. In all cases studied so far, maximal value of $\int \alpha$ on region of linear size L goes as $L^{k-1} \ln(L)$.

Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

Thanks!

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Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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Framing the question
Some puzzles
Pattern-Equivariant Cohomology
Cohomology solves transport
Answers to puzzles
Conjectures, partial results and open problems

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- Zeit für Bier?