Homeomorphisms of Tiling Spaces

Lorenzo Sadun

University of Texas

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Outline

1. Results (joint with Antoine Julien)

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The classification theorem

In the category of minimal, uniquely ergodic FLC tiling spaces, homeomorphisms $h : \Omega \to \Omega'$ of a given tiling space Ω are classified, up to homotopy and MLD, by an open subset of the Čech cohomology $\check{H}^1(\Omega, \mathbb{R}^d)$.

The decomposition theorem

In the same category, every homeomorphism is a composition of

- A "weak translation" $\psi : \Omega \to \Omega$, homotopic to the identity map.
- A shape change homeomorphism $h_{sc} : \Omega \to \Omega''$, and
- An MLD equivalence $\phi : \Omega'' \to \Omega'$.

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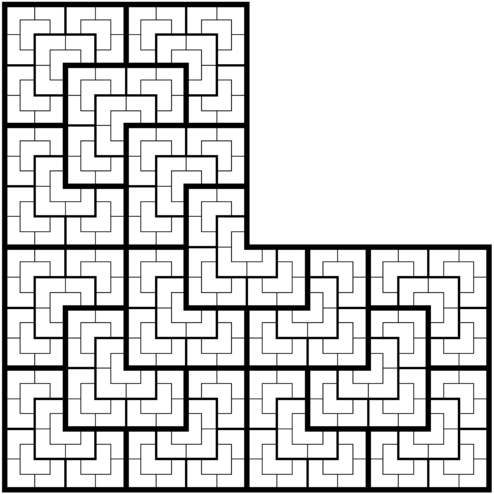
What is an FLC tiling?

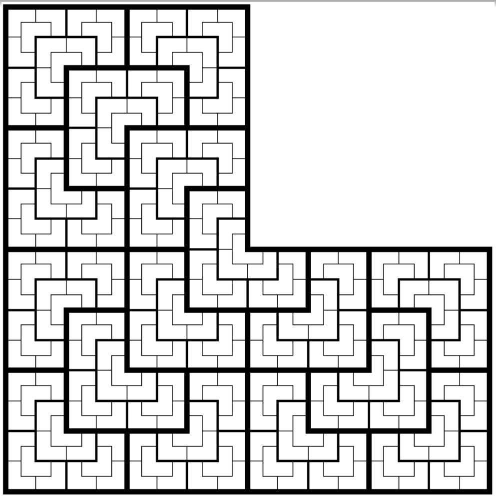
An FLC tiling is a collection of regions ("tiles") in $\mathbb{R}^d$ such that

- Each tile is a closed topological disk $t = \overline{t^0}$, together with a label. WLOG, tiles are labeled polyhedra.
- Any two tiles with the same label must be translates of one another.
- Tiles intersect only on their boundaries.
- The union of all the tiles is all of $\mathbb{R}^d$.
- There are only finitely many tile labels.
- There are only finitely many ways for two tiles to overlap. (No continuous shears.)

A pretty tiling



A different pretty tiling



The tiling metric

- Two tilings T and T' are ϵ-close if they agree on $B_{1/\epsilon}(0)$, up to translation by up to ϵ.
- Behavior of T and T' near ∞ is irrelevant.
- Metric is *not* translation-invariant, but resulting topology *is* translation-invariant. (Convergence on compact sets)

Definition of a tiling space

A **tiling space** Ω is a non-empty compact translation-invariant set of tilings, using the topology induced by the tiling metric. We usually also assume:

- Non-periodicity: If $T - x = T$, then $x = 0$.
- Minimality: If $T \in \Omega$, then $\Omega = \overline{\{T - x\}}$.
- Unique ergodicity, aka existence of uniform patch frequencies.

Local topology of Ω

ϵ-neighborhood N of T is $B \times C$, where B is a ball in $\mathbb{R}^d$ and C is a Cantor set. To get $T' \in N$,

- Erase T outside of $B_{1/\epsilon}(0)$.
- Translate by up to ϵ (this gives B).
- Fill in near ∞. Infinitely many discrete choices gives C.

Global topology of Ω

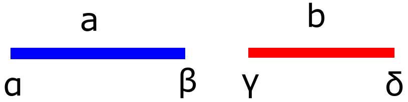
- Ω is the inverse limit of a sequence of CW-complexes, describing a tiling out to greater and greater distances.
- Ω is connected but not path-connected. There are uncountably many path-components.
- Many constructions, all very similar. Here's Franz Gähler's.

The Anderson-Putnam complex

Describes how to place a single tile containing the origin.

- Which type of tile? Finite choice.
- Where is the origin in that tile? Need copy of that tile.
- What if 0 is on the boundary of two or more tiles? Identify common edges/faces/etc.

The AP complex of the Fibonacci tiling



- *ab* is legal, so $\beta = \gamma$.
- *ba* is legal, so $\delta = \alpha$.
- *aa* is legal, so $\alpha = \beta$.
- This forces $\gamma = \delta$, even though *bb* isn't legal.
- Γ is a figure 8, with all four vertices identified.

Collaring

Relabel tiles by the pattern of nearest neighbors. For Fibonacci:

- $A_1 = (a)a(b)$
- $A_2 = (b)a(a)$
- $A_3 = (b)a(b)$
- $B_1 = (a)b(a)$

$\ldots abaababa \ldots$ becomes $\ldots A_? B_1 A_2 A_1 B_1 A_3 B_1 A_? \ldots$

Gähler's construction

- Γ^n = AP complex with n-collared tiles.
- A point in Γ^n is instructions for laying a tile at the origin plus n rings around it. (OK, tile touching the origin plus $n-1$ rings if origin is on the boundary.)
- $\rho_n : \Gamma^n \to \Gamma^{n-1}$ is forgetful.
- $\Omega = \varprojlim(\Gamma^n, \rho_n)$ = consistent instructions for tiling all of $\mathbb{R}^d$.

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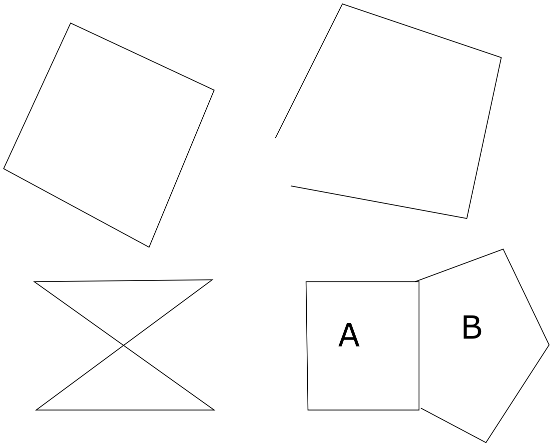
Weak translations

- Pick a continuous map $f : \Omega \to \mathbb{R}^d$.
- $\psi(T) = T - f(T)$
- Homotopic to the identity map $(\Psi_s(T) = T - sf(T))$
- Homeomorphism as long as f is Lipschitz with constant < 1.

Shape changes

- Assign vector $\vec{v}(e)$ to every edge e of every face.
- Corresponding edges of same tile type get same vector.
- $\sum \vec{v}(e) = 0$ for edges around a tile.
- Consistency: If two tile share an edge, both tiles give same vector (up to sign).
- $\vec{v}$ is closed $\mathbb{R}^d$ valued 1-cochain on AP complex.
- Can also work with n-collared tiles.

Same thing in pictures



Shape change maps

- Replace each edge e with $\vec{v}(e)$.
- Use piecewise linear map on each tile type to fix origin.
- Preserves combinatorics of how tiles meet.
- Changes the geometry
- Avoiding crossings is an open condition. Maps are parametrized, up to MLD, by $\check{H}^1(\Omega, \mathbb{R}^d)$.

Mutual Local Derivability (MLD)

- $\ell : \Omega \to \Omega'$ is a **local derivation** if
  - ℓ is a factor map (commutes with translations), and
  - There is a radius R such that $B_1(0) \cap \ell(T)$ is determined exactly by $B_R(0) \cap T$.
  - Analogue of "sliding block code"
- $m : \Omega \to \Omega'$ is MLD if both m and m^{-1} are LD.

Examples of MLD maps

- Translation by a fixed vector $\vec{x}_0$.
- Relabeling tiles ($\ldots abba \ldots$ vs. $\ldots 0110 \ldots$)
- Splitting or rejoining tiles. E.g. replace every a tile with two smaller tiles labeled a_1 and a_2.
- Land grabs. E.g. move the boundary of every ab meeting 1 cm to the right. This affects *collared* tiles, making $a(b)$ bigger and $(a)b$ smaller.

That's all there is!

Restatement of main result (part 2): Any homeomorphism $h : \Omega \to \Omega'$ of minimal, uniquely ergodic FLC tiling spaces can be decomposed as $h = \phi \circ h_{SC} \circ \Psi$, where

- Ψ is a weak translation
- h_{SC} is a shape change
- ϕ is an MLD equivalence.

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The usual algebraic topology invariants don't work

- Ω is connected but not path connected.
- Each path component is contractible.
- Invariants that probe one path component at a time, like π_n and singular (co)homology, are useless.
- Need something that measures connections at ∞.

Čech cohomology

- Definition of $\check{H}^*(X)$ is complicated (and unnecessary!)
- If X is a CW complex, then $\check{H}^*(X)$ is isomorphic to every other cohomology theory (singular, cellular, simplicial).
- If $X = \varprojlim \Gamma^n$, then $\check{H}^*(X) = \varinjlim H^*(\Gamma^n)$.
- $\check{H}^*(\Omega, A) = \varinjlim \check{H}^*(\Gamma^n, A) = \varinjlim H^*(\Gamma^n, A)$.

The dyadic solenoid

- $\mathbb{S}_2 = \varprojlim(S^1, \rho)$ where $\rho(z) = z^2$.
- $H^0(S^1) = \mathbb{Z}$, $\rho_0^* =$ identity, $\check{H}^0(\mathbb{S}_2) = \mathbb{Z}$.
- $H^1(S^1) = \mathbb{Z}$, $\rho_1^* = \times 2$, $\check{H}^1(\mathbb{S}_2) = \mathbb{Z}[1/2]$.

Pattern-equivariant (PE) cohomology (Kellendonk-Putnam)

- Start with tiling T, not with tiling *space* Ω.
- PE function on T with radius R is a function whose value at x only depends on $B_R(0) \cap (T - x)$.
- "f is PE" means "f is PE with some finite but unspecified radius R."
- $C^k =$ PE functions on k-cells of T.
- $\delta_k : C^k \to C^{k+1}$, $\delta\alpha(c) := \alpha(\partial c)$. $\delta^2 = 0$.
- $H^*_{PE,T} = \dfrac{Ker(\delta)}{Im(\delta)}$

PE = Čech

Theorem (Kellendonk-Putnam, S-)

If Ω is a minimal space of FLC tilings, $T \in \Omega$, and A is an abelian group, then $H^*_{PE,T}$ with values in A is isomorphic to $\check{H}^*(\Omega, A)$.

What are PE cochains?

$$\begin{aligned} C^k &= \{\text{PE k-cochains on } T\} \\ &= \varinjlim(\text{k-cochains on } \Gamma^n) \end{aligned}$$

Proof

$$
\begin{aligned}
H^k_{PE,T} &= \frac{Ker\delta_k}{Im\delta_{k-1}} \\
&= \frac{\varinjlim(\text{Ker of } \delta_k \text{ on } \Gamma^n)}{\varinjlim(\text{Im of } \delta_{k-1} \text{ on } \Gamma^n)} \\
&= \varinjlim \left(\frac{\text{Ker of } \delta_k \text{ on } \Gamma^n}{\text{Im of } \delta_{k-1} \text{ on } \Gamma^n} \right) \\
&= \varinjlim H^k(\Gamma^n, A) = \varinjlim \check{H}^k(\Gamma^n, A) \\
&= \check{H}^k(\Omega, A)
\end{aligned}
$$

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The fundamental shape class

- α(edge) = actual vector along that edge.
- $\delta\alpha(\text{face}) = \alpha(\partial(\text{face})) = 0$.
- α is a closed PE 1-cochain with values in $\mathbb{R}^d$.
- Defines a cohomology class $\mathcal{F}(\Omega) = [\alpha]$.

The cohomology class of a homeomorphism

- Given homeomorphism $h : \Omega \to \Omega'$,
- Class of h is $[h] = h^*[\mathcal{F}(\Omega')] \in \check{H}^1(\Omega, \mathbb{R}^d)$.
- Pullback is in Čech cohomology.
- Can also do pullback of PE cochain, but only after homotoping h so that it preserves transversals and so preserves PE functions.

Example: Shape changes for Fibonacci

- Let Ω be a Fibonacci tiling space whose tiles a and b have length ℓ_a and ℓ_b. $\mathcal{F}(\Omega) = \ell_a i_a + \ell_b i_b$.
- Let Ω' be Fibonacci tilings whose tiles A and B have length L_A and L_B. $\mathcal{F}(\Omega') = L_A i_A + L_B i_B$.
- h is shape change. $h^*(i_A) = i_a$, $h^*(i_B) = i_b$.
- $[h] = L_A i_a + L_B i_b$.

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Statement of the theorem

Theorem

If $h_1 : \Omega \to \Omega_1$ and $h_2 : \Omega \to \Omega_2$ are homeomorphisms of minimal FLC tiling spaces, and if $[h_1] = [h_2]$, then there is a commutative diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{h_1} & \Omega_1 \\ \Psi \big\downarrow & & \big\downarrow \phi \\ \Omega & \xrightarrow{h_2} & \Omega_2 \end{array}$$

where Ψ is a weak translation and ϕ is an MLD equivalence.

Proof sketch, part 1

- Let $\alpha_{1,2}$ represent $[h_{1,2}]$.
- By continuity, big patch P at origin of T determines patch P_1 (resp P_2) at origin of $h_1(T)$ (resp $h_2(T)$), up to small translation.
- If P occurs at points x and y in T, displacement between corresponding P_i's in $h_i(T)$'s is approximately $\displaystyle\int_x^y \alpha_i$.
- Since $\alpha_2 = \alpha_1 + \delta f$, integrals are the same for big enough P.
- P_1's in $h_1(T)$ almost line up with P_2's in $h_2(T)$.

Proof sketch, part 2

- By applying a weak translation, we can change "almost line up" to "line up exactly".
- "Line up exactly" plus homeomorphism implies MLD equivalence (possibly with gigantic radius).

Proof sketch, part 2

- By applying a weak translation, we can change "almost line up" to "line up exactly".
- "Line up exactly" plus homeomorphism implies MLD equivalence (possibly with gigantic radius).
- Lots of technical details. I did say "sketch"!

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The RS map with forms

- Can represent classes in $H^*_{PE,T}(\Omega, \mathbb{R}^d)$ with vector-valued PE forms on $\mathbb{R}^d$. (Original construction)
- E.g., represent $\mathcal{F}(\Omega)$ by $\displaystyle\sum_{j=1}^{d} dx^j \otimes e_j$
- Define $RS(\alpha) = \displaystyle\int_\Omega \alpha d\mu$.
- Need unique ergodic measure μ.

What does $RS([h])$ tell us?

- Ergodic theorem: Averaging α over a big region in $\mathbb{R}^d$ gives $RS(\alpha)$, up to small error.
- Convergence is uniform, thanks to unique ergodicity.
- If $\alpha \in H^1(\Omega, \mathbb{R}^d)$, $RS(\alpha)$ is a linear transformation.
- $RS([h])$ gives distortion induced on each leaf, at large scale. (Requires technical estimates)

Theorem 1

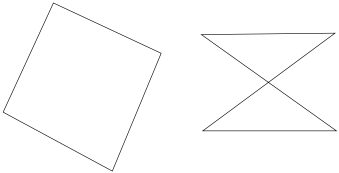
Claim: $RS([h])$ is invertible. Sketch of proof:

- If $RS([h])(v) = 0$, then h collapses things in v direction at large scale.
- But then h^{-1} stretches things by arbitrarily large factor over long distances.
- Contradicts uniform continuity of h^{-1}.

Theorem 2

Claim: If $\alpha \in \check{H}^1(\Omega, \mathbb{R}^d)$ and $RS(\alpha)$ is invertible, then there is a shape change whose class is α.

- Danger is map folding things over at small scale.



- Not a problem if α is shape change done by pointwise almost constant form.
- Take any representative of α and convolve with very wide bump function. Becomes pointwise close to $RS(\alpha)$.

Proof of decomposition theorem

- Given homeomorphism $h : \Omega \to \Omega'$
- Let $\alpha = [h]$.
- By Theorem 1, $RS(\alpha)$ is invertible.
- By Theorem 2, there is a shape change h_{sc} with $[h_{sc}] = \alpha$.
- By structure theorem, h and h_{sc} are related by ...

The punch line

$$\begin{array}{ccc} \Omega & \xrightarrow{h} & \Omega' \\ \Psi \downarrow & & \downarrow \phi \\ \Omega & \xrightarrow{h_{sc}} & \Omega'' \end{array}$$

$h = \phi^{-1} \circ h_{sc} \circ \Psi$ as claimed.

The punch line

$$\begin{array}{ccc} \Omega & \xrightarrow{h} & \Omega' \\ \Psi \downarrow & & \downarrow \phi \\ \Omega & \xrightarrow{h_{sc}} & \Omega'' \end{array}$$

$h = \phi^{-1} \circ h_{sc} \circ \Psi$ as claimed. (Modulo ϕ vs ϕ^{-1})).

The classification theorem, revisited

- Set of possible $[h]$'s is RS^{-1}(invertible matrices).
- RS^{-1}(invertible matrices) is an open subset of $\check{H}^1(\Omega, \mathbb{R}^d)$.
- This open subset parametrizes all possible shape changes, up to homotopy and MLD.

The classification theorem, revisited

- Set of possible $[h]$'s is RS^{-1}(invertible matrices).
- RS^{-1}(invertible matrices) is an open subset of $\check{H}^1(\Omega, \mathbb{R}^d)$.
- This open subset parametrizes all possible shape changes, up to homotopy and MLD.
- By the decomposition theorem, the same subset parametrizes all homeomorphisms, up to homotopy and MLD.

Open questions

Let G be the group of self-homeomorphisms of Ω.

- How does G act on RS^{-1}(invertible matrices)?
- How does G act on all of $\check{H}^1(\Omega, \mathbb{R}^d)$?
- $\{$Tiling spaces homeomorphic to $\Omega\}/$MLD $=$ RS^{-1}(invertible matrices)$/$(Action of G).