Topological Groups with Strong Disconnectedness Properties

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A space X is

- maximal if it is crowded (has no isolated points) and admits no finer crowded topology $\iff$ no point is limit for two disjoint subsets $\iff$ a subset is open iff it is crowded;
- extremally disconnected if any two disjoint open subsets of X have disjoint closures ($\Leftrightarrow$ the closure of any open set is open);
- basically disconnected if the closure of any cozero set in X is open;
- an F-space if any two disjoint cozero sets are completely (= functionally) separated in X;
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- a P-space if any (co)zero set is clopen ($\Leftrightarrow$ any G_δ-set is open).

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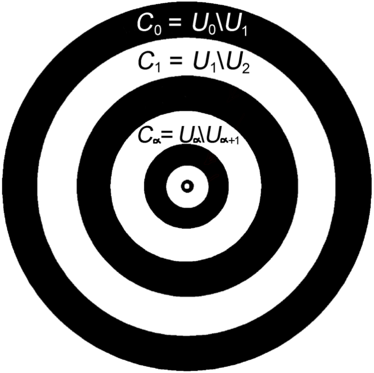
Fact

$\exists$ *extremally disconnected P-space* $\implies$ $\exists$ *measurable cardinal.*

Indeed, let X be a zero-dimensional space, and let $x \in X$ be such that $x \notin \overline{U} \cap \overline{V}$ for any disjoint open $U, V \subset X$.

We construct a strictly decreasing sequence $(U_\alpha)_{\alpha<\kappa}$ of clopen neighborhoods of x such that $x \notin \operatorname{Int} \bigcap_{\alpha<\kappa} U_\alpha$. Clearly, κ is limit.



For each $\alpha \in \kappa$ let $C_\alpha = U_\alpha \setminus U_{\alpha+1}$. The sets C_α are clopen and pairwise disjoint, and for any neighborhood $V \ni x$ $\exists \alpha \in \kappa$ such that $V \cap C_\alpha \neq \varnothing \implies$ the sets $\{\alpha \in \kappa : V \cap C_\alpha \neq \varnothing\}$ for all neighborhoods V of X form an ultrafilter. It is countably complete if x is a P-point.

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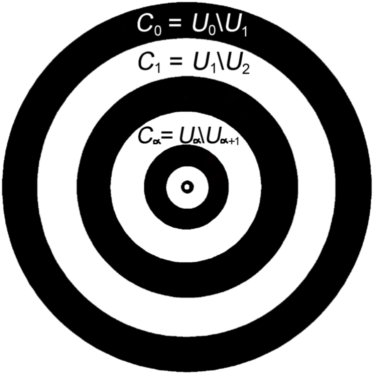

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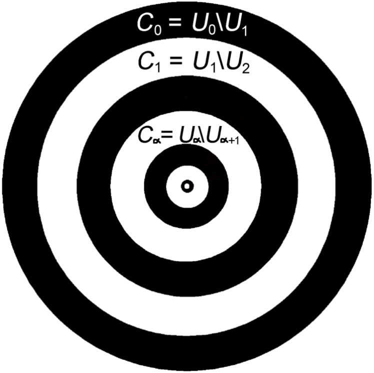

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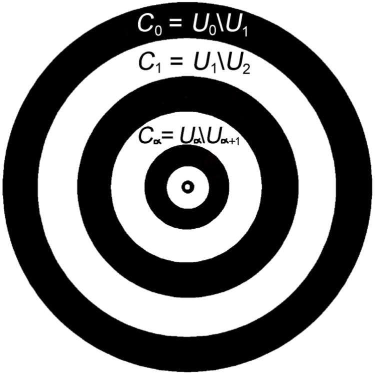

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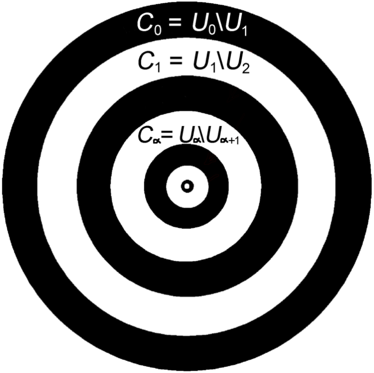

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Extremally disconnected compact spaces = projective objects in the category of compact spaces and continuous maps (that is, if E, X, and Y are compact spaces, E is extremally disconnected, $f\colon E \to X$ is continuous, and $g\colon Y \to X$ is continuous and surjective, then there exists a continuous $\varphi\colon E \to Y$ such that $f = g \circ \varphi$:

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X is extremally disconnected $\iff$ the Boolean algebra $\mathrm{CO}(X)$ of clopen sets is complete and forms a base of the topology of X;

X is basically disconnected $\iff$ $\mathrm{CO}(X)$ is countably complete and forms a base of the topology of X;

X is a strongly zero-dimensional F-space $\iff$ $\mathrm{CO}(X)$ is weakly countably complete and forms a base of the topology of X ("weakly countably complete" means that if $A, B \subset CO(X)$ are countable and $a \wedge b = 0$ for $a \in A$, $b \in B$, then $\exists\, c \in CO(X)$ such that $a \leqslant c$ and $b \leqslant \neg c$ for all $a \in A$, $b \in B$).

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For each $f \in C(X)$ $P_f = \bigcap$(all minimal prime ideals containing f).

- X is extremally disconnected $\iff$ every intersection of ideals P_f is principal.
- X is a P-space $\iff$ every nonunit element of $C(X)$ is a zero divisor and every ideal P_f is principal $\iff$ every prime ideal in $C(X)$ is principal.
- X is basically disconnected $\iff$ every ideal P_f is principal.
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A strongly disconnected X is homogeneous $\implies$ any compact $K \subset X$ is finite.

Problem

Do there exist (in ZFC) strongly disconnected topological groups not being P-spaces?

MALYKHIN:

- $\mathfrak{p} = \mathfrak{c} \implies \exists$ nondiscrete maximal topological groups
- Any extremally disconnected topological group has an open Boolean subgroup (Boolean = all elements are of order 2)
  $h\colon G \to G$, $x \mapsto x^{-1}$, is a self-homeomorphism. Frolík's theorem $\implies$ $U = \operatorname{Fix} f$ is a neighborhood of 1. If $V \ni 1$ and $V \cdot V \subset U$, then $\langle V \rangle$ is a Boolean subgroup.
- Any maximal topological group contains a countable neighborhood of 1.
  In any Boolean group $B(X) = [X]^{<\omega}$ with uncountable neighborhoods of 1 the sets $\{F \in B(X)\colon$ the power of 2 in the prime factorization of $|F|$ is even/odd$\}$ are dense and disjoint.

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Maximal groups: *The nonexistence of maximal nondiscrete groups is consistent with ZFC.*

$\exists$ countable nondiscrete extremally disconnected group $\implies$ $\exists$ rapid filter on ω [Reznichenko+S].

Extremally disconnected groups: (a) *The nonexistence of countable nondiscrete extremally disconnected groups is consistent with ZFC.* (b) $\exists$ *measurable cardinals* $\implies$ $\exists$ *nondiscrete extremally disconnected groups.*
(c) $\exists$ *a nondiscrete extremally disconnected group with linear (generated by subgroups) topology* $\implies$ $\exists$ *either a measurable cardinal or such a group of cardinality* $\leqslant 2^{\omega}$*.*

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(Consistently) exists a basically disconnected group G, not a P-space, containing no open Boolean subgroups:

$G = G_1 \times G_2$, where G_1 is a countable nondiscrete extremally disconnected group and G_2 is an arbitrary nondiscrete P-group [Comfort+Hindman+Negrepontis].

Fact (Hart+Vermeer)

If K is a compact finite-dimensional F-space and $\phi\colon K \to K$ is a continuous injective map, then the fixed-point set of ϕ is a P-set of K, that is, every G_δ-set containing it is a neighbourhood of it.

Theorem

Any Abelian F-group G with $\dim_0 G < \infty$ and $\psi(G) \leqslant \omega$ contains an open Boolean subgroup with the same properties.

Corollary

The existence of an Abelian topological F-group G with $\dim_0 G < \infty$ and $\psi(G) \leqslant \omega$ is equivalent to the existence of a nondiscrete Boolean topological group with the same properties.

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Theorem

Any basically disconnected Abelian topological group either is a P-space or has a nondiscrete topological quotient of countable pseudocharacter containing an open basically disconnected Boolean subgroup.

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The existence of an Abelian basically disconnected group which is not a P-space is equivalent to the existence of a nondiscrete Boolean basically disconnected group of countable pseudocharacter.

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$F_G(X)$, $A_G(X)$, and $B_G(X)$ are the free, free Abelian, and free Boolean topological groups of a $T_{3\frac{1}{2}}$-space X in the sense of Graev: given an arbitrary fixed point $x_0 \in X$, the group $F_G(X)$ is the unique topological group with $1 = x_0$ containing X as a subspace and such that any continuous map of X to any topological group G that takes x_0 to the identity element of G can be extended to a continuous homomorphism $F_G(X) \to X$. $F_G(X)$ does not depend on x_0. Any free topological group in the sense of Markov is a free topological group in the sense of Graev.

The free Abelian and Boolean topological groups are defined in a similar way. Algebraically, the free Boolean group generated by a set X is $[X]^{<\omega}$ with the operation of symmetric difference.

Fact

The free Abelian topological group $A_G(X)$ is the topological quotient of $F_G(X)$ by the commutator subgroup, and the free Boolean topological group $B_G(X)$ is the topological quotient of $A_G(X)$ by the subgroup $A_G(2X)$ of squares.

Fact

If G is a topological group, H is its subgroup, and G/H is the quotient space of left or right cosets, then the canonical quotient map $G \to G/H$ is open.

Fact

Extremal disconnectedness, basic disconnectedness, and the property of being an F'-space are preserved by open continuous maps.

Indeed, for such a map $f\colon X \to Y$ and any $A \subset Y$, we have $\overline{A} = f(\overline{f^{-1}(A)})$.

Fact

For any $T_{3\frac{1}{2}}$-space X, the following conditions are equivalent:

1. X *is a* P*-space;*
2. $F_G(X)$ *is a* P*-space;*
3. $A_G(X)$ *is a* P*-space;*
4. $B_G(X)$ *is a* P*-space.*

Theorem

1. *For a $T_{3\frac{1}{2}}$-space X, the following conditions are equivalent:*
   - *the free topological group of X is an F'-space,*
   - *the free Abelian topological group of X is an F'-space,*
   - *X is a P-space.*
2. *The existence of a free Boolean topological F'-group which is not a P-space is equivalent to the existence of a selective ultrafilter on ω.*

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THANK YOU