Exercise Sheet 2, Proseminar Stochastic Processes
Winter Semester 2016-17, 250069 PS

Exercise 1 *Given a Markov chain on a state space E, let $P = (p_{ij})_{i,j\in E}$ denote a transition matrix, and $P^n = (p_{ij}^{(n)})_{i,j\in E}$ its n-th power. Show the following:*

1. *If C is a communicating class of a finite Markov chain, show that there is $n \in \mathbb{N}$ such that $p_{ij}^{(n)} > 0$ for all $i, j \in C$.*
2. *If C is a closed communicating class, then $(p_{ij})_{i,j\in C}$ is a probability matrix.*
3. *Every finite Markov chain has at least one closed communicating class.*
4. *Find an example of a Markov chain without closed communicating class.*

Exercise 2 *Given is the transition matrix*

$$P = (p_{ij})_{i,j=1}^4 = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

1. *What are the communication classes?*
2. *Compute the expected return time μ_k for each state.*
3. *For which i, j holds: $\lim_n p_{i,j}^{(n)} \to 1/\mu_j$?*

Exercise 3 *A chess-king moves on a 3×3 chess-board, taking each of its possible moves (a king can move to each of the neighbouring square, straight or diagonally) with equal probability. What are the limit visit frequencies to each of the squares?*

You can try for the other chess-pieces too.

Exercise 4 (Example how Kolmogorov Extension Theorem works) *Let $\Omega = \{(\omega_n)_{n\geq 0} : \omega_n \in \{-1, 1\}\}$ be the probability space of a stochastic process of coinflips with a fair coin. Let $[e_0 \dots e_{k-1}] = \{(\omega_n)_{n\geq 0} \in \Omega : \omega_n = e_n,\ 0 \leq n < k\}$ be a k-cylinder. Since the coin is fair, $\mathbb{P}([e_0 \dots e_{k-1}]) = (\frac{1}{2})^k$ for every choice of $e_0, \dots e_{k-1}, \in \{-1, 1\}$.*

Let

$$K_{k,m} = \bigcup \Big\{ [e_0, \dots, e_{k-1}] : |\frac{1}{k} \#\{0 \leq i < k : e_i = -1\} - \frac{1}{2}| \leq \frac{1}{m} \Big\}.$$

It can be computed that the complement $\mathbb{P}(K_{k,m}^c) \leq \frac{2m}{\sqrt{2\pi}} e^{-\frac{2k}{m^2}}$.

1. *Let $\mathcal{F}$ be the σ-algebra generated by all cylinder sets. Show that*
$$L = \{(\omega_n)_{n\geq 0} \in \Omega : \lim_{k\to\infty} \frac{1}{k} \#\{0 \leq i < k : \omega_i = -1\} = \frac{1}{2}\}$$
belongs to $\mathcal{F}$.
2. *Show that $\mathbb{P}(L) = 1$.*