Limit cykels

Def. A solution $\gamma(t)$ of an ODE $\dot{x} = F(x)$ in $\mathbb{R}^d$ is periodic of period $T > 0$ if $\gamma(t) = \gamma(t+T)$ $\forall t$ and T is indeed the smallest positive number with this property

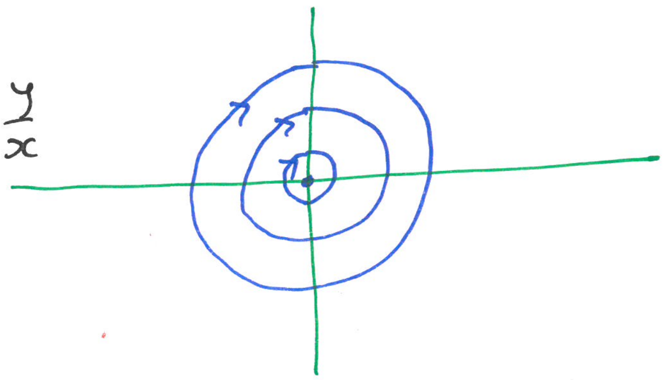
NB: Stationary points are not called periodic.

Example $\circledast \begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ has only periodic solutions encircling a single stationary point at $(0,0)$

NB: This is the harmonic oscillator $\ddot{x} + \omega^2 x = 0$ with $\omega = 1$.

In polar coordinates

$r^2 = x^2 + y^2 \qquad \varphi = \arctan \frac{y}{x}$



$\circledast$ takes the form

$$\begin{cases} \dot{r} = 0 \\ \dot{\varphi} = -1 \end{cases}$$

with solution: $r = \text{Const} \qquad \varphi(t) = \varphi_0 - t$.

Def A limit cykel is a periodic solution of an ODE that is isolated from other periodic solutions and that are attracting in forward (or repelling in backward) time.

They are easy to produce with polar coordinates

$$\begin{cases} \dot r = r - r^2 \\ \dot\varphi = 1 \end{cases} \Rightarrow \dot r = 0 \text{ if } r = 0 \text{ (stationary pt)} \text{ or } r = 1 \text{ (periodic solution and also stable, so a limit cykel)}$$

Back to Cartesian coordinates

$$\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix} \Rightarrow \begin{cases} \dot x = \dot r\cos\varphi - r\sin\varphi\,\dot\varphi = (1-r)x - y. \\ \dot y = \dot r\sin\varphi + r\cos\varphi\,\dot\varphi = (1-r)y + x. \end{cases}$$

$$\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - (x^2+y^2)\begin{pmatrix} x \\ y \end{pmatrix} \qquad (*)$$

Stationary point $(0,0)$ has eigenvalues $1 \pm i\sqrt{2}$ unstable spiral

At $r=1$ $(*)$ reduces to $\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as for harmonic oscillator.

Hilbert's 16^{th} problem is about limit cykels:
For polynomial ODEs in $\mathbb{R}^2$, of degree n, what is the maximal possible number $H(n)$ of limit cykels.

$H(1) = 0$, $H(2) \geq 4$, $H(3) \geq 13$
precise value unknown.

Van der Pol equations

Van der Pol (1889–1959) was engineer at Philips labs between the wars, later professor in Delft. He worked on early transistors and vacuum tubes

(*) $$\ddot{x} + x = \underbrace{\varepsilon(1-x^2)\dot{x}}_{\text{friction term in oscillator}}$$

$x^2 < 1$: negative friction

$x^2 > 1$: positive friction

Rewrite to Initial Value Problem:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \varepsilon(1-x^2)\,y \end{cases} = F\begin{pmatrix} x \\ y \end{pmatrix}, \quad x(0) = \bar{x}, \; y(0) = 0$$

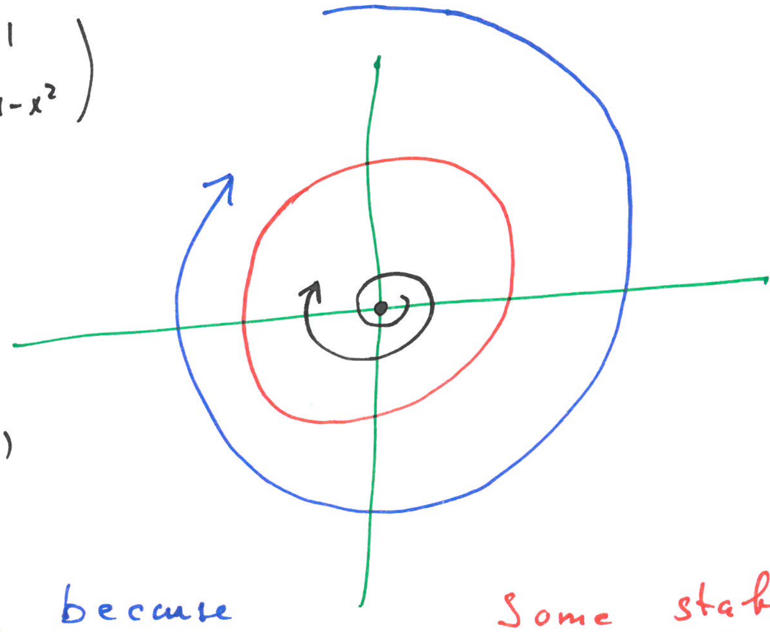
$$DF_{(x,y)} = \begin{pmatrix} 0 & 1 \\ -1-2\varepsilon xy & \varepsilon(1-x^2) \end{pmatrix}$$

$$DF_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix}$$

has eigenvalues

$$\lambda = \frac{\varepsilon}{2} \pm \sqrt{\frac{\varepsilon^2}{4} - 1}$$

so spiral source at $(0,0)$



But for $\bar{x}$ large, because of positive friction, solution should spiral inwards to the origin

Some stable limit cycle should be in between.

Van der Pol eq. for small ε : Multiscale analysis

Ansatz: Solution $x(t) = \sum_{k \geq 0} \varepsilon^k x_k(t)$.

zeroth power
ε-terms not present

$$(*_0) \begin{cases} \ddot{x}_0 + x_0 = 0 \\ \underbrace{x_0(0) = \bar{x}, \ \dot{x}_0 = 0}_{\text{initial values}} \end{cases} \Rightarrow \begin{cases} x_0(t) = \bar{x} \cos t \\ \dot{x}_0(t) = -\bar{x} \sin t \end{cases}$$

1st power
all terms linear in ε

$$(*_1) \begin{cases} \varepsilon \ddot{x}_1 + \varepsilon x_1 = \varepsilon(1 - x_0^2)\dot{x}_0 \\ x_1(0) = \dot{x}_1(0) = 0 \end{cases}$$

initial value $\bar{x}$ is incorporated in x_0

Insert results of x_0, $\dot{x}_0$ and cancel ε.

$$\ddot{x}_1 + x_1 = \bar{x}(\bar{x}^2 \cos^2 t - 1) \sin t$$

$$= \underbrace{\bar{x}\left(\frac{\bar{x}^2}{4} - 1\right) \sin t}_{} + \frac{\bar{x}^2}{4} \sin 3t$$

Use trig. form $\cos^2 t \sin t = \frac{1}{4}(\sin t + \sin 3t)$

driving force with same frequency as "natural" frequency (i.e. of $x_0(t)$) $\Rightarrow$ expect resonance

solution increases over time. Ansatz looses validity

Solution of $(*_1)$

$$x_1(t) = \frac{\bar{x}}{2} \underbrace{\left(\frac{\bar{x}^2}{4} - 1\right)}_{} (\sin t - t \cos t) + \frac{\bar{x}^3}{32}(3 \sin t - \sin 3t)$$

However, if $\bar{x} = 2$, then the resonance terms disappears
Ansatz remains valid. Limit cykel near $\bar{x} = 2$

$$\cancel{u\frac{r_0}{v}|\sin\alpha_0| \qquad z = \left(\frac{v_0\sin\alpha}{u}\right) dz = -\frac{v_0\sin\alpha_0}{u}}$$

3a

$$+ \quad x_1(t) = \frac{\bar{x}}{2}\left(\frac{\bar{x}^2}{4}-1\right)(\sin t - t\cos t) + \frac{\bar{x}^3}{32}(3\sin t - \sin 3t)$$

$$\dot{x}_1(t) = \frac{\bar{x}}{2}\left(\frac{\bar{x}^2}{4}-1\right)(\cancel{\cos t} - \cancel{\cos t} + t\sin t) + \frac{\bar{x}^3}{32}(3\cos t - 3\cos 3t)$$

$$+ \quad \ddot{x}_1(t) = \frac{\bar{x}}{2}\left(\frac{\bar{x}^2}{4}-1\right)(\sin t + t\cos t) + \frac{\bar{x}^3}{32}(-3\sin t + 9\sin 3t)$$

$$\ddot{x}_1(t) + x_1(t) = \frac{\bar{x}}{2}\left(\frac{\bar{x}^2}{4}-1\right) 2\sin t + \frac{\bar{x}^3}{32}\, 8\sin 3t$$

$$= \bar{x}\left(\frac{\bar{x}^2}{4}-1\right)\sin t + \frac{\bar{x}^3}{4}\sin 3t.$$

✓

Verify solution of ($*_1$)

$$+\quad x_1(t) = \frac{\bar{x}}{2}\left(\frac{\bar{x}^2}{4}-1\right)\ \sin t - t\cos t \quad + \frac{\bar{x}^3}{32}\left(3\sin t - \sin 3t\right)$$

$$\dot{x}_1(t) = \frac{\bar{x}}{2}\left(\frac{\bar{x}^2}{4}-1\right)\left(\cancel{\cos t} - \cancel{\cos t} + t\sin t\right) + \frac{\bar{x}^3}{32}\left(3\cos t - 3\cos 3t\right)$$

$$+\quad \ddot{x}_1(t) = \frac{\bar{x}}{2}\left(\frac{\bar{x}^2}{4}-1\right)\left(\sin t + t\cos t\right) + \frac{\bar{x}^3}{32}\left(-3\sin t + 9\sin 3t\right)$$

$$\ddot{x}_1(t) + \dot{x}_1 = \frac{\bar{x}}{2}\left(\frac{\bar{x}^2}{4}-1\right) 2\sin t + \frac{\bar{x}^3}{32}\cdot 8\sin 3t$$

$$= \bar{x}\left(\frac{\bar{x}^2}{4}-1\right)\sin t + \frac{\bar{x}^3}{4}\sin 3t = \text{RHS of } (*_1)\ \checkmark$$

Alternative Approach

Kinetic Energy $\quad E = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2$

"Lie derivative" $\quad \frac{dE}{dt} = x\dot{x} + \dot{x}\ddot{x} = \dot{x}(x+\ddot{x})$ by ($*$)

$$= \varepsilon\dot{x}^2(1-x^2)$$

amplitude

Start with $x_0(t) = A\cos t$ and substitute:

$$\frac{dE}{dt} = \varepsilon A^2 \sin^2 t\ (1 - A^2\cos^2 t)$$

If $x(t)$ is periodic, then integrating $\frac{dE}{dt}$ over one period should give 0:

$$0 = \int_0^{2\pi}\frac{dE}{dt}\,dt = \varepsilon A^2\int_0^{2\pi}\sin^2 t\,(1-A^2\cos^2 t)\,dt$$

$$= \varepsilon A^2\int_0^{2\pi}\sin^2 t\,dt - \varepsilon A^4\int_0^{2\pi}\sin^2 t\cos^2 t\,dt$$

$$= \varepsilon A^2\pi\left(1 - \frac{A^2}{4}\right)$$

so amplitude $A = 2$.

$\sin t\cos t = \frac{1}{2}\sin 2t$

Van der Pol eq. for large ε: Relaxation oscillation

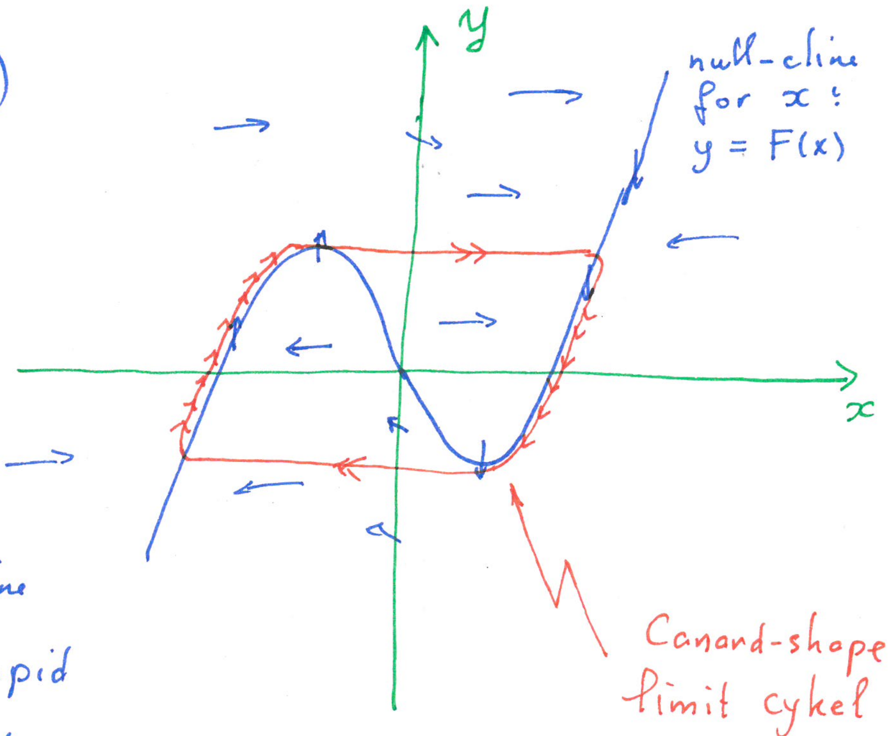
$$(*) \qquad \ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$$

Lienard transformation $\quad w = \varepsilon\underbrace{\left(\frac{x^3}{3} - x\right)}_{F(x)} + \dot{x}$

$$\begin{cases} \dot{x} = w - \varepsilon F(x) & \leftarrow \text{from Lienard} \\ \dot{w} = -x & \leftarrow \text{from } (*) \end{cases}$$

Next $\quad y = w/\varepsilon \qquad$ Recall ε is large, so y is small.

$$\begin{cases} \dot{x} = \varepsilon(y - F(x)) \\ \dot{y} = -\frac{1}{\varepsilon}x \end{cases}$$



Away from the null-cline $|\dot{x}|$ is very large: rapid horizontal movement, and null-cline is attracting

Relaxation oscillation: slow movement along null-cline. Quick "release" at local extrema of null-cline.