TRANSVERSE ENTROPY of FOLIATED IFS's

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SUMTOPO 22

Setting

$(X, \mathcal{F})$ – a compact foliated space (manifold, matchbox etc.) modelled transversely on a metric space Z $(= \mathbb{R}^q$, Cantor set etc.)

$\mathfrak{F} = \langle f_1, \dots, f_m \rangle$ – a finitely generated pseudo(semi-)group (IFS) of continuous local foliated maps $f : D_f \to R_f$, D_f, $R_f \subset X$:

$$f \in \mathfrak{F} \;\Leftrightarrow\; \forall_{x \in D_f} \exists_U \exists_{n \in \mathbb{N}} \exists_{k_1, \dots k_n} \; x \in U \subset D_f, \; f|U = f_{k_1} \circ \dots f_{k_n}|U$$

$\mathcal{A}$ – a finite foliated atlas on X

$\mathfrak{H} = \mathfrak{H}_{\mathcal{A}}$ – the holonomy pseudogroup of $\mathcal{F}$ defined on Z by $\mathcal{A}$

$\mathfrak{G} = \mathfrak{G}_{\mathfrak{F}, \mathfrak{H}}$ – the finitely generated pseudo(semi-)group of maps $g \circ f \circ h$ with $f \in \mathfrak{F}$, $g, h \in \mathfrak{H}$

Pseudo(semi-)groups

Definition

Given an arbitrary topological space Z, the set $\mathfrak{G}$ of continuous maps $g : D_g \to Z$, D_g being open in Z, is said to be a pseudosemigroup (PSG) whenever

(i) $g, h \in \mathfrak{G} \implies g \circ h \in \mathfrak{G}$,

(ii) $g \in \mathfrak{G},\ U \subset D_g \implies g|U \in \mathfrak{G}$,

(iii) *$g : D_g \to Z$ is continuous and D_g is covered by such open sets U that $g|U \in \mathfrak{G}$ for all $U \Longrightarrow g \in \mathfrak{G}$.*

If moreover

(iv) $\mathrm{id}_X \in \mathfrak{G}$, *then $\mathfrak{G}$ is called a pseudomonoid*

and if all $g \in \mathfrak{G}$ are homeomorphisms between open sets D_g and R_g such that

(v) $g \in \mathfrak{G} \Rightarrow g^{-1} \in \mathfrak{G}$,

then $\mathfrak{G}$ is a pseudogroup.

Examples

- *Given a smooth manifold M and $r \geq 1$, all local C^r maps $g : D_g \to M$, $D_g \subset M$, form a PSG (in fact, a pseudomonoid).*
- *Given a foliated space $(X, \mathcal{F})$ and $r \geq 1$, all the local foliated maps $g : D_g \to X$, $D_g \subset X$, being C^r-smooth along the leaves form a PSG (a pseudomonoid).*
- *Given a Riemannian manifold $(M, \langle, \cdot, \cdot\rangle)$ all the local isometric, quasi-isometric, (quasi-)conformal maps form PSG's.*
- *And many more.*

Proposition

For any set $\mathfrak{G}_0$ of local maps $g : D_g \to Z$ there exists the smallest PSG $\mathfrak{G}$ containing $\mathfrak{G}_0$.

Proof.

$\mathfrak{G}$ can be defined either as the intersection of all the PSG's containing $\mathfrak{G}_0$ or as the set of all the maps $g : D_g \to Z$ satisfying the following:

$$\forall_{x\in D_g} \exists_U \exists_{n\in\mathbb{N}} \exists_{g_1,\dots g_n \in \mathfrak{G}_0} \; x \in U \subset D_g, \;\; g|U = g_1 \circ \dots g_n|U.$$

$\square$

Definition

The PSG of the above Proposition is said to be generated by $\mathfrak{G}_0$. PSG's generated by finite sets are said to be finitely generated.

Let $\mathfrak{G}$ and $\mathfrak{H}$ be PSG's on spaces X and Y, respectively, and Φ a set of homeomorhisms $\phi : D_\phi \to R_\phi$ between open sets of X and Y such that $\bigcup_{\phi \in \Phi} D_\phi = X$.

Definition

Φ is said to be a *morphism* of $\mathfrak{G}$ into $\mathfrak{H}$, $\Phi : \mathfrak{G} \to \mathfrak{H}$, whenever $\phi \circ g \circ \psi^{-1} \in \mathfrak{H}$ for all $g \in \mathfrak{G}$, $\phi, \psi \in \Phi$. Φ is an *isomorphism* between $\mathfrak{G}$ and $\mathfrak{H}$ when $\Phi : \mathfrak{G} \to \mathfrak{H}$ and $\Phi^{-1} = \{\phi^{-1}; \phi \in \Phi\} : \mathfrak{H} \to \mathfrak{G}$.

From foliated maps to PSG's

Now: $\mathfrak{F}_1 = \{F_1, \dots F_k\}$ – a finite set of foliation preserving maps of a foliated space $(X, \mathcal{F})$ into itself,
$\mathfrak{F}_k$, $k \in \mathbb{N}$ – the set of all $F_{i_k} \circ \cdots \circ F_{i_1}$'s, $\mathfrak{F} = \bigcup_{k \in \mathbb{N}} \mathfrak{F}_k$.

Certainly: $\mathfrak{F}$ is a semigroup, a monoid when $\mathrm{id}_X \in \mathfrak{F}_1$ (if so, $\mathfrak{F}_k \subset \mathfrak{F}_{k+1}$ for any $k \in \mathbb{N}$).

Next: $\mathcal{A} = \{\phi_a; a \in A\}$ – a foliated atlas, D_a – the domain of ϕ_a. $T_a \subset D_a$ – transversals, T – the disjoint union of all the T_a's, a complete transversal: any leaf L of $\mathcal{F}$ intersects T.

$\forall F \in \mathfrak{F}$, $a, b \in A$: $F_{b,a} = \pi_b \circ F \circ \iota_a$, where $\iota_a : T_a \to D_a$ – the embedding, $\pi_b : D_b \to T_b$ – the projection along the plaques.
Certainly: domains D_{ba} of F_{ba}'s are open in T_a's and
$F_{ca} = F_{cb} \circ F_{ba}$ whenever defined.

Finally: $\mathfrak{G}_0 = \{F_{ba}; a, b \in A\}$ generates a PSG $\mathfrak{G}(\mathfrak{F}, \mathcal{A})$ on T.
Certainly again, $\mathrm{id}_X \in \mathfrak{F}_1 \implies \mathfrak{G}(\mathfrak{F}, \mathcal{A})$ – a pseudomonoid.

Remark

The similar construction can be performed for any finitely generated PSG $\mathfrak{F}$ of local foliation preserving maps $F : D_F \to X$ defined on open domains $D_F \subset X$.

On the next slide:

$\mathfrak{G}$ is an arbitrary PSG generated by a finite set $\mathfrak{G}_0$.

Definition

For any $\epsilon > 0$ and $k \in \mathbb{N}$, points x and y of X are said to be (k, ϵ)-*separated* whenever $d(h(x), h(y)) \geq \epsilon$ for some $h \in \mathfrak{G}_k$. The set $Z \subset X$ is (k, ϵ)-*separated* whenever any two points x and y of Z, $x \neq y$, are (k, ϵ)-separated. Since X is compact, all the (k, ϵ)-separated sets are finite and one can put

$$N(k, \epsilon, \mathfrak{G}_0) = \max \#\{Z \subset T, Z \text{ is } (k, \epsilon) - \text{separated}\},$$

$$N(\epsilon, \mathfrak{G}_0) = \limsup_{k \to \infty} \frac{1}{k} \cdot \log N(k, \epsilon, \mathfrak{G}_0),$$

$$\mathcal{E}(\mathfrak{G}, \mathfrak{G}_0) = \lim_{\epsilon \to 0} N(\epsilon, \mathfrak{G}_0).$$

The quantity $\mathcal{E}(\mathfrak{G}, \mathfrak{G}_0)$ is called the *topological entropy* of $\mathfrak{G}$ (*w. r. t.* $\mathfrak{G}_0$).

Definition

Given a foliated space $(X, \mathcal{F})$, a finitely generated PSG $\mathfrak{F}$ of foliation preservig maps and a foliated atlas $\mathcal{A}$, the quantity

$$\mathcal{E}^{\pitchfork} = \mathcal{E}(\mathfrak{G}(\mathfrak{F}, \mathcal{A}), \mathfrak{F}_1)$$

is called the *transverse entropy* of $\mathfrak{F}$ (w. r. t. $\mathcal{F}, \mathcal{A}$ and a generating set $\mathfrak{F}_1$).

Proposition

If the two distance functions d_1 and d_2 on X are Lipschitz equivalent, $c^{-1}d_1 \leq d_2 \leq c \cdot d_1$ for some constant $c \geq 1$, then the corresponding entropies $\mathcal{E}_1$ and $\mathcal{E}_2$ are equal. Consequently, if $(X, \mathcal{F})$ is a closed foliated manifold and the distances d_1 and d_2 on a transversal T arise from Riemannian structures g_1 and g_2 on X, then the corresponding transverse entropies $\mathcal{E}_1^{\pitchfork}$ and $\mathcal{E}_2^{\pitchfork}$ are equal.

Proposition

If $\mathfrak{G}_0$ and $\mathfrak{G}'_0$ are two finite sets generating a PSG $\mathfrak{G}$, and for any $g \in \mathfrak{G}_0$ and any $x \in D_g$ there exist $m \in \mathbb{N}$, a neighbourhhod U of x and a member h of $\mathfrak{G}'_m$ such that $g = h$ on U, then

$$\mathcal{E}(\mathfrak{G}, \mathfrak{G}_0) \leq m_0 \cdot \mathcal{E}(\mathfrak{G}', \mathfrak{G}'_0).$$

for some $m_0 \in \mathbb{N}$.

Corollary

For any finitely generated PSG $\mathfrak{F}$ of foliation preserving maps and any two foliated atlases $\mathcal{A}$ and $\tilde{\mathcal{A}}$ on a compact foliated space $(X, \mathcal{F})$ the corresponding transverse entropies $\mathcal{E}^{\pitchfork} = \mathcal{E}(\mathfrak{G}(\mathfrak{F}, \mathcal{A}), \mathfrak{F}_1)$ and $\tilde{\mathcal{E}}^{\pitchfork} = \mathcal{E}(\mathfrak{G}(\mathfrak{F}, \tilde{\mathcal{A}}), \mathfrak{F}_1)$ satisfy

$$c \cdot \mathcal{E}^{\pitchfork} \leq \tilde{\mathcal{E}}^{\pitchfork} \leq C \cdot \mathcal{E}^{\pitchfork}$$

for some $c, C > 0$.

Therefore, *with a bit of care, one can distinguish between PSG's of foliated maps of zero transverse entropy and those of positive transverse entropy.*

Definition

A (Borel, probability) measure μ on Z is said to be $\mathfrak{G}$-invariant whenever $\mu(h^{-1}(Y)) = \mu(Y)$ for any $h \in \mathfrak{G}$ and any measurable $Y \subset D_h$.
For a PSG $\mathfrak{F}$ of foliation preserving maps a foliated atlas $\mathcal{A}$, $\mathfrak{G}(\mathfrak{F}, \mathcal{A})$-invariant measures are said to be transversely invariant (TIM).

Simple examples show, that non-trivial measures invariant by transformations of a group, semigroup, therefore of a PSG, needn't exist.

Theorem

If $\mathfrak{G}$ has vanishing entropy, then $\mathfrak{G}$-inavariant Borel probability measures exist. Therfore, if $\mathcal{E}^{\pitchfork}(\mathfrak{F}) = 0$, then Borel probability TIM's for $\mathfrak{F}$ do exist.

The proof is analogous to that for holonomy pseudogroups in

E. Ghys, R. Langevin, P. W., *Entropie geometrique des feuilletages*, Acta Math. **160** (1988), 105 – 142

and for general pseudogroups in Section 4.5 of

P. W., *Dynamics of Foliations, Groups and Pseudogroups*, Birhkhäuser 2004.

Existence of TIM's - the proof (1)

Proof. Let $S(k,\epsilon)$ be the family of all (k,ϵ)-separated subsets of Z. Given a non-negative continuous function f on Z put

$$\Lambda_{k,\epsilon}(f) = \frac{1}{N(k,\epsilon)} \cdot \sup\left\{ \sum_{y\in Y} f(y);\, Y \in S(k,\epsilon) \right\}$$

The functionals $\Lambda_{k,\epsilon}$ are non-negatively homogeneous, subadditive, monotonic, normaliezed and bounded: $\Lambda_{k,\epsilon}(f) \le \sup f$. Since $\mathcal{E}(\mathfrak{G}) = 0$, there exist sequences $\epsilon_n \to 0$ and $\mathbb{N} \ni k_n \to \infty$ such that the corresponding functionals Λ_{k_n,ϵ_n} and $\Lambda_{k_n+1,\epsilon_n}$converge to functionals Λ and Λ_1 which are non-negatively homogeneous, subadditive, monotonic and normalized: $\Lambda(1) = \Lambda_1(1) = 1$.

Moreover

$$\Lambda(f_1 + f_2) = \Lambda(f_1) + \Lambda(f_2) \text{ and } \Lambda_1(f_1 + f_2) = \Lambda_1(f_1) + \Lambda_1(f_2)$$

whenever $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \emptyset$. Put

$$\mu(K) = \inf\{\Lambda(f); 0 \leq f \leq 1, f|K \equiv 1\}$$

and

$$\mu_1(K) = \inf\{\Lambda_1(f); 0 \leq f \leq 1, f|K \equiv 1\}.$$

Following the proof of Riesz Reprezentation Theorem, see

W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1966,

one can show that μ and μ_1 extend to Borel probability measures such that $\mu = \mu_1$ is $\mathfrak{G}$-invariant. $\square$

Theorem (GLW)

If the entropy of a foliation $\mathcal{F}$ of a manifold vanishes, then the Godbillon-Vey class $GL(\mathcal{F}) = 0 \in H^3(M)$.

In

M. V. Losik, *A generalization of manifold and its characteristic classes*, Funktsional'nyi Analiz i Ego Prilozheniya, **24** (1990), 29 –37,

V. Bazaikin, A. Galayev, P. Gumenyuk, *Non-diffeomorphic Reeb foliations and modified Godbillon-Vey class*, Math. Z. **300** (2022), 1335 – 1349,

a characteristic (diffeo-not-homeo invariant!) class GVL($\mathcal{F}$) for codimension-1 foliations has been introduced (and studied). GVL$(\mathcal{F}) \in H^3(S_2(M/\mathcal{F}))$, where $S_2(M/\mathcal{F})$ is the second order frame bundle of the leaf space $M/\mathcal{F}$ considered as the generalised manifold in the Losic sense.

Since

(1.1) there exists a natural linear map $H^3(S_2(M/\mathcal{F}) \to H^3(M)$ which sends $GVL(\mathcal{F})$ onto $GV(\mathcal{F})$,

(1.2) all the Reeb foliations of S^3 have zero entropy while some of them have non-zero GVL-class [BGG],

(2) Losik generalized manifolds correspond to orbit spaces of peseudugroups,

one can ask the following (and more)

Questions

(1) When does the condition $\mathcal{E}^{\pitchfork}(\mathcal{F}) = 0$ imply $GVL(\mathcal{F}) = 0$?

(2) Can one extend the Losik notion of a manifold to orbit spaces of arbitrary PSG's and provide GLV-classes for reasonable PSG's?

Terminus

Finally let me express my best thanks:
to the organizers for the invitation and
to participants for attending my talk.

Figure : Grimming that I hope to see soon