Self-similarity of substitution tiling semigroups

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Joint work with Jamie Walton (Nottingham)

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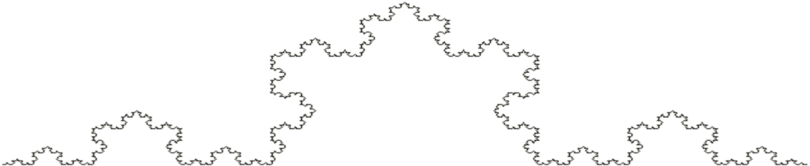

Plan

1. I. Tilings and their properties
2. II. Kellendonk's punctured hull and topological Markov shifts
3. III. Self-similar inverse semigroup actions
4. IV. The limit space (there and back again)

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II. Kellendonk's punctured hull

For each prototile $p \in \mathcal{P}$, choose a point in its interior and label it $x(p)$. Extend the punctures to every tiling T' in Ω by the rule

$$t \in T' \text{ and } t = p + y \qquad \Longrightarrow \qquad x(t) = x(p) + y.$$

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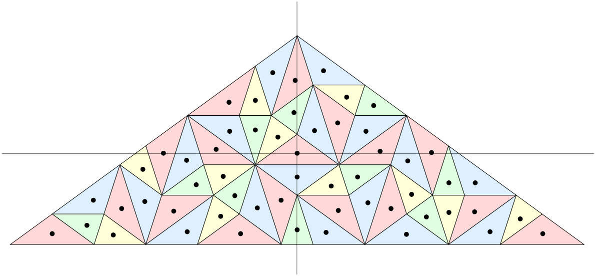

The discrete hull of a tiling

A neighbourhood base for the topology of Ω_{punc} is given by: For P a patch of tiles in some tiling in Ω_{punc} and $t \in P$ define

$$U(P,t) := \{ T \in \Omega_{\mathrm{punc}} \mid P - x(t) \subset T \}$$

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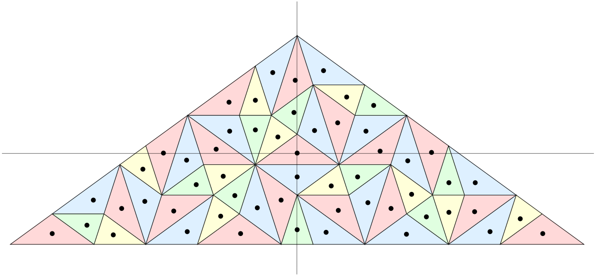

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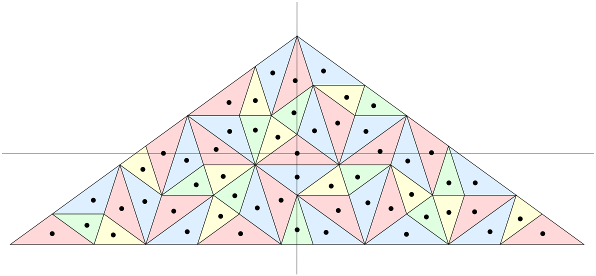

Lemma (Kellendonk): The sets $U(P,t)$ are clopen, and Ω_{punc} is a Cantor set.

The substitution graph and discrete hull

For $p \in \mathcal{P}$ and $t \in \omega(p)$ we call (t, p) a supertile extension, and denote the set of such supertile extensions by $\mathcal{S}$.

Often $\mathcal{S}$ can be identified with all pairs $(a, b) \in \mathcal{P}^2$ for which $a \in \omega(b)$.

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Given a supertile extension $e = (t, p) \in \mathcal{S}$, we denote $r(e) := t$ (considered as a prototile in $\mathcal{P}$) and $s(e) := p$. We construct the substitution graph G with vertex set $\mathcal{P}$, edge set $\mathcal{S}$ and source and range maps $s, r \colon \mathcal{S} \to \mathcal{P}$.

The substitution graph and discrete hull

The substitution graph gives rise to a topological Markov shift

$$\mathcal{F} := \{e_0 e_1 e_2 \cdots \mid s(e_i) = r(e_{i+1})\},$$

with left shift $\sigma : \mathcal{F} \to \mathcal{F}$ defined by $\sigma(e_0 e_1 e_2 \cdots) = e_1 e_2 \cdots$.

There is a natural topology on $\mathcal{F}$ whose basis consists of clopen cylinder sets of all infinite strings starting with some given finite initial string.

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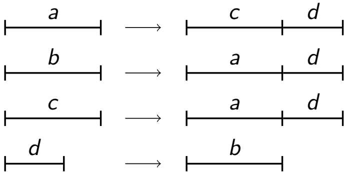
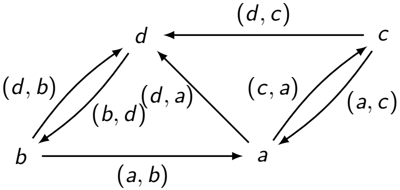
We obtain a homeomorphism between the Markov shift and the punctured hull:

$$\tau : \mathcal{F} \xrightarrow{\cong} \Omega_{punc}$$

The Fibonacci tiling

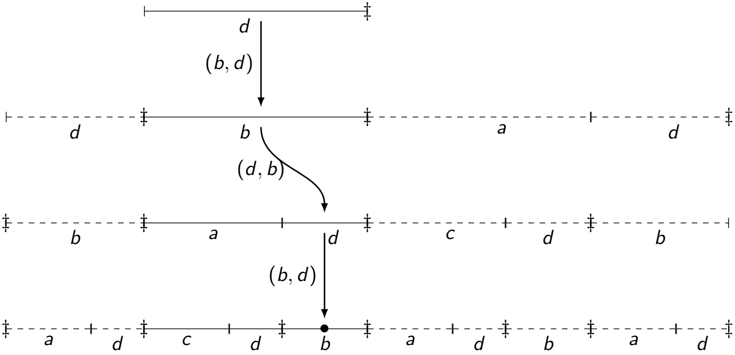
The (border forcing) Fibonacci substitution has four prototiles that substitute as follows:





The substitution graph is on the right, and paths in the graph correspond to punctured tilings...

The Fibonacci tiling



A graphical representation of the partial tiling formed by the word $w = (b, d)(d, b)(b, d)\ldots$. Tile lengths are increased by the golden ratio at each increasing level.

III. Self-similar inverse semigroup actions

A semigroup S is an inverse semigroup if for each $s \in S$ there exists a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$.

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An action of an inverse semigroup S on a set X is a homomorphism $\pi : S \to \mathcal{I}(X)$. If the homomorphism π is fixed, we usually write $g \cdot x$ for $\pi_g(x)$.

Tiling inverse semigroups (Kellendonk)

A doubly pointed patch $[b, P, a]$ is given by a finite patch P and tiles $a, b \in P$, where we take the tuple (b, P, a) up to translation equivalence. Let $\mathcal{T}$ be the set of all doubly pointed patches along with a 'zero element' $0 \in \mathcal{T}$.

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Let $[d, Q, c]$, $[b, P, a] \in \mathcal{T}$ be two doubly pointed patches which, without loss of generality (by translating each, if necessary) have $x(b) = x(c)$. If P and Q agree on any tiles with intersecting interiors and $P \cup Q$ is a valid patch, then we define

$$[d, Q, c][b, P, a] = [d, P \cup Q, a].$$

Otherwise, we define $[d, Q, c][b, P, a] = 0$. Any product with 0 is defined as 0.

We call $\mathcal{T} = (\mathcal{T}, \cdot)$ the tiling semigroup.

Tiling inverse semigroups (Kellendonk)

The tiling semigroup $\mathcal{T}$ naturally acts by partial bijections on the discrete hull Ω_{punc}, where a doubly pointed patch $g = [b, P, a]$ has domain $U(P, a) \subset \Omega_{punc}$ and codomain $U(P, b) \subset \Omega_{punc}$.

The action is by translation: If $T \in U(P, a) \subset \Omega_{punc}$, then

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Note that our notation mirrors function composition: the element $[b, P, a]$ has 'in tile' a and 'out tile' b, with a product of elements $[d, Q, c][b, P, a]$ interpreted as applying the right then the left-hand term, and requiring that the intermediate tiles b and c agree.

Self-similar inverse semigroup actions

Definition (Barthold, Grigorchuk and Nekrashevych)

Let $\mathcal{F}$ be a topological Markov chain over an alphabet X. An inverse semigroup G acting on $\mathcal{F}$ is called self-similar if for every $g \in G$ and $x \in X$ there exist $y_1, \ldots, y_k \in X$ and $h_1, \ldots, h_k \in G$ such that the sets $\mathrm{dom}(h_i)$ are disjoint, $\bigcup_{i=1}^{k} x\mathrm{dom}(h_i) = x\mathcal{F} \cap \mathrm{dom}(g)$, and for every $xw \in \mathcal{F}$ we have

$$g \cdot xw = y_i(h_i \cdot w), \qquad (1)$$

where i is such that $w \in \mathrm{dom}(h_i)$.

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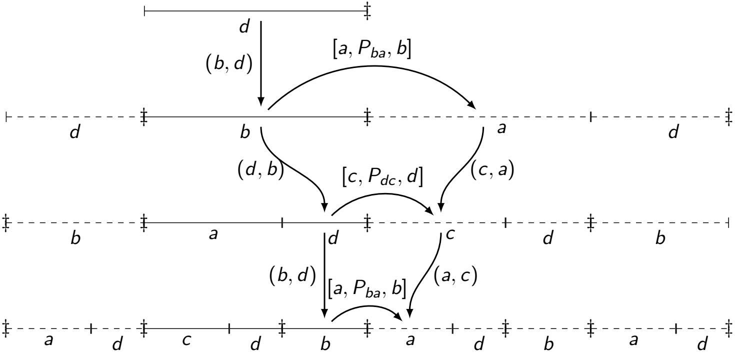
Theorem (Walton-W)

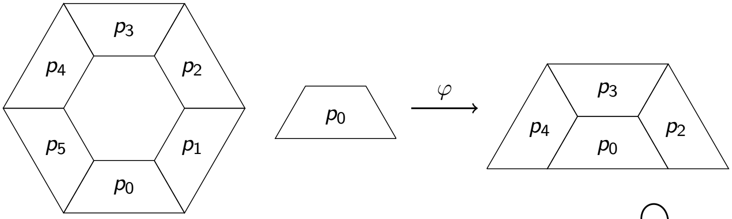
The tiling semigroup $\mathcal{T}$ of a substitution tiling defines a self-similar inverse semigroup action on the Markov shift $\mathcal{F}$.

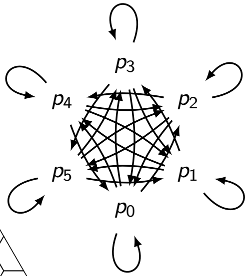
Back to the Fibonacci example

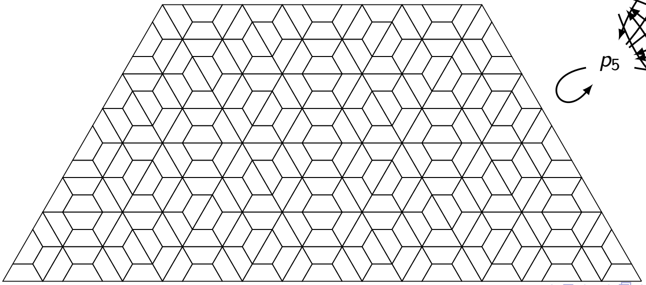


$$\begin{aligned}[a, P_{ba}, b] \cdot (b, d)(d, b)(b, d)z &= (a, c)\left([c, P_{dc}, d] \cdot (d, b)(b, d)z\right) \\ &= (a, c)(c, a)\left([a, P_{ba}, b] \cdot (b, d)z\right)\end{aligned}$$

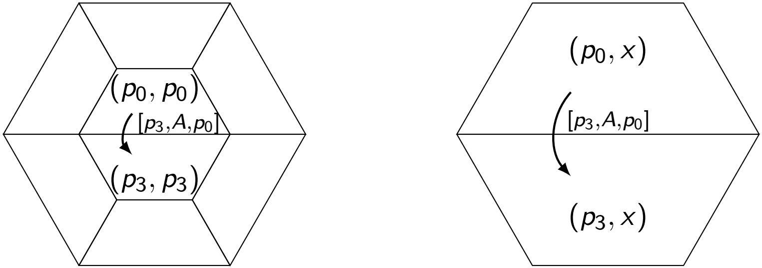
Example: the half hex tiling







Half-hex con't



An illustration of the self-similar relation

$$[p_3, A, p_0] \cdot (p_0, p_0)w = (p_3, p_3)\ [p_3, A, p_0] \cdot w.$$

IV. The limit space (there and back again)

Let G be a finite graph with associated topological Markov shift $\mathcal{F}$ (the right-infinite, left-pointing paths). We define

$$\mathcal{F}^- := \{\cdots e_{-3}e_{-2}e_{-1} \mid r(e_i) = s(e_{i-1})\};$$

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Two elements $x = \cdots e_{-3}e_{-2}e_{-1}$ and $y = \cdots f_{-3}f_{-2}f_{-1} \in \mathcal{F}^-$ are called asymptotically equivalent with respect to the action of the semigroup G if there is a sequence (g_n) of G, with $\{g_n\} \subseteq G$ finite, and some $w \in \mathcal{F}$ so that for each $n \in \mathbb{N}$ the element

$$g_n \cdot (e_{-n} \cdots e_{-3}e_{-2}e_{-1}w) \in \mathcal{F}$$

has initial string of n terms given by $f_{-n} \cdots f_{-3}f_{-2}f_{-1} \in \mathcal{F}^n$. In this case we write $x \sim_{\mathrm{ae}} y$. We define the asymptotic equivalence relation $\sim$ on $\mathcal{F}^-$ to be the equivalence relation generated by $\sim_{\mathrm{ae}}$.

The limit space

The limit space $\mathcal{J}$ of a self-similar semigroup action is defined as the quotient space $\mathcal{F}^- / \sim$. The shift map $\sigma \colon \mathcal{F}^- \to \mathcal{F}^-$, given by $\cdots e_{-3}e_{-2}e_{-1} \mapsto \cdots e_{-4}e_{-3}e_{-2}$ induces a map $\sigma \colon \mathcal{J} \to \mathcal{J}$. We denote its inverse limit by

$$\Omega := \varprojlim(\mathcal{J} \overset{\sigma}{\leftarrow} \mathcal{J} \overset{\sigma}{\leftarrow} \mathcal{J} \overset{\sigma}{\leftarrow} \cdots). \tag{2}$$

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The Anderson–Putnam complex of a tiling is the compact Hausdorff topological space formed through taking the transitive closure of gluing together prototiles in all ways their translations can be adjacent in a tiling.

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Theorem (Walton-W)

Suppose $(\mathcal{T}, \mathcal{F})$ is a self-similar inverse semigroup associated with a recognisable substitution ω. The limit space $\mathcal{J}$ is homeomorphic to the Anderson–Putnam complex of the substitution, and the inverse limit Ω in (2) is conjugate to the continuous hull Ω_ω.

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