The eventual hyperbolic dimension of entire functions

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Workshop on ergodic theory and holomorphic dynamics
1 October 2015

- An important class of invariant measures, are the measures which are absolutely continuous with respect to Lebesgue measure.
- Absolutely continuous invariant measures can be contructed for a lot of real one dimensional systems such as real quadratic mappings.
- Although, this is only possible for only a few cases in the complex one dimensional setting.

Conformal measures for transcendental functions

Definition

Let f be holomorphic, let $t > 0$. The measure μ is *t-conformal* for f, if, for any measurable set E on which f in injective,

$$\mu(f(E)) = \int_E |f'|^t d\mu.$$

Existence of conformal measures have been showed for many of the rational maps.
Conformal measures have been constructed only for some transcendental functions:

- For some exponential maps (Urbański-Zdunik),
- For a large class of meromorphic functions (Mayer-Urbański "balanced growth condition"),
- Other examples (?)

Dynamics near ∞

The behavior of the function near infinity plays an essential role in the above constructions of conformal measures.

Definition (Escaping set)

Let f be an entire function, its escaping set is

$$I(f) = \{z : f^n(z) \to \infty\}.$$

Unlike polynomials, the Julia set of transcendental functions intersects any neighborhood of ∞.

The eventual dimension

The Hausdorff dimension of the escaping set of a transcendental entire function can be any value in $[1, 2]$.

Theorem (Rempe-Gillen-Stallard)

For all $d \in [1, 2]$ there exists a function f in class $\mathcal{B}$ such that $\mathrm{HD}(I(f)) = d$.

In their proof they use the *eventual dimension*.

Definition (Eventual dimension)

$$\mathrm{edim}(f) = \lim_{R\to\infty} \mathrm{HD}\left(\{z : \forall n \geq 0, |f^n(z)| \geq R\}\right).$$

The above examples have $\mathrm{edim}(f) = \mathrm{HD}(I(f))$.

Hyperbolic sets

Definition (Hyperbolic set)

The compact set $X \subset J(f)$ is a hyperbolic set if $f(X) \subset X$ and there exists $n > 0$ and $\kappa > 1$ such that

$$|f^{n\prime}(z)| \geq \kappa$$

for all $z \in X$.

Definition (Hyperbolic dimension)

The **hyperbolic dimension** $\text{hypdim}(f)$ of a function f is the supremum of the dimensions of the hyperbolic sets of f.

Remark

The exponent of the conformal measures is equal to the hyperbolic dimension.

Eventual hyperbolic dimension

Definition (Eventual hyperbolic dimension)

$$\text{ehypdim}(f) = \lim_{R\to\infty} \sup\{\text{HD}X : X \subset \mathbb{C}\backslash D_R \text{ hyperbolic set}\}$$

Proposition

The exponential maps and the functions satisfying Mayer-Urbański's "balanced growth" condition all have $\text{ehypdim}(f) = 1$.

Poincaré functions (1)

- Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic.
- A **Poincaré function** L is a lineariser of a repelling periodic point of f, that is, $L : \mathbb{C} \to \mathbb{C}$ is holomorphic and satisfies the Schröder equation:

$$\begin{array}{ccc} \mathbb{C} & \overset{w \mapsto \rho w}{\longrightarrow} & \mathbb{C} \\ L \downarrow & & \downarrow L \\ \mathbb{C} & \underset{f^{\circ p}}{\longrightarrow} & \mathbb{C} \end{array}$$

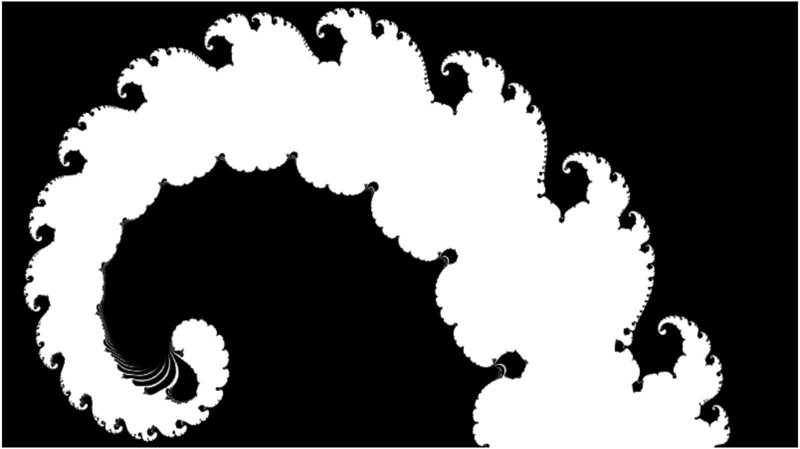
  where p is the period and ρ the multiplier, and L sends 0 on one of the points of the periodic cycle.
- A choice of $L'(0)$ uniquely determine L.

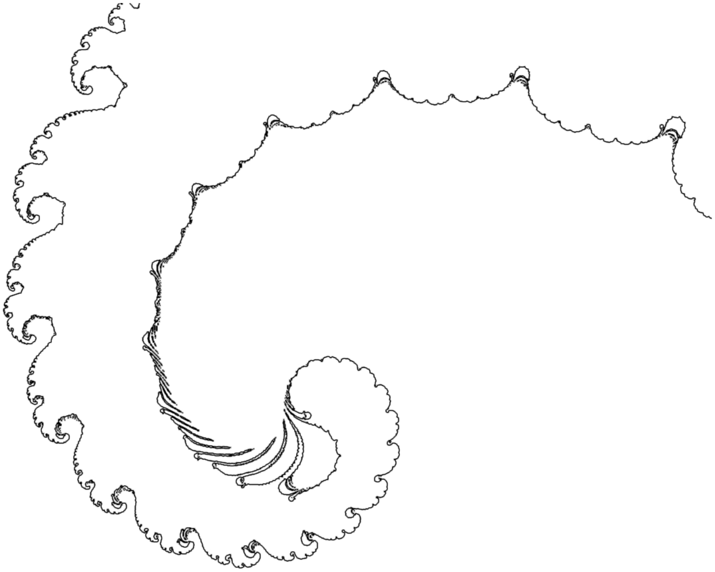
Poincaré functions (2)

Example

Let $f(z) = z^d$, with $d \geq 2$. Then the normalised Poincaré function associated to the fixed point $z = 1$ is $L(z) = e^z$.

- If f is entire, then L is entire.
- The singular set of a Poincaré function L for a repelling fixed point of an entire function f is the postsingular set of f.
- In particular, if the map f is hyperbolic, the function L belongs to the class $\mathcal{B}$.
- Poincaré functions of polynomials have finite positive order.





linearisers of polynomials (1)

Theorem

1. *Let P be a polynomial and let L be a Poincaré function of a repelling periodic point of P. Then*

$$\mathrm{ehypdim}(L) \geq \mathrm{hypdim}(P).$$

2. *If moreover the polynomial is hyperbolic and if its Julia set is connected, then*

$$\mathrm{ehypdim}(L) = \mathrm{hypdim}(P) = \mathrm{HD}(J(P)) < 2.$$

Remark

The second part holds indeed for any topological Collet-Eckmann polynomial.

Corollary

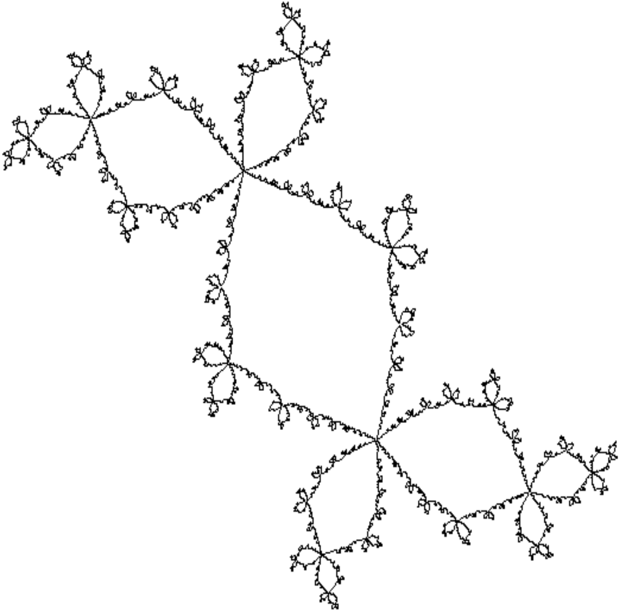
If P is as above and is not conjugated to a Chebychev polynomial or a monomial, then there are Poincaré functions L of P such that:

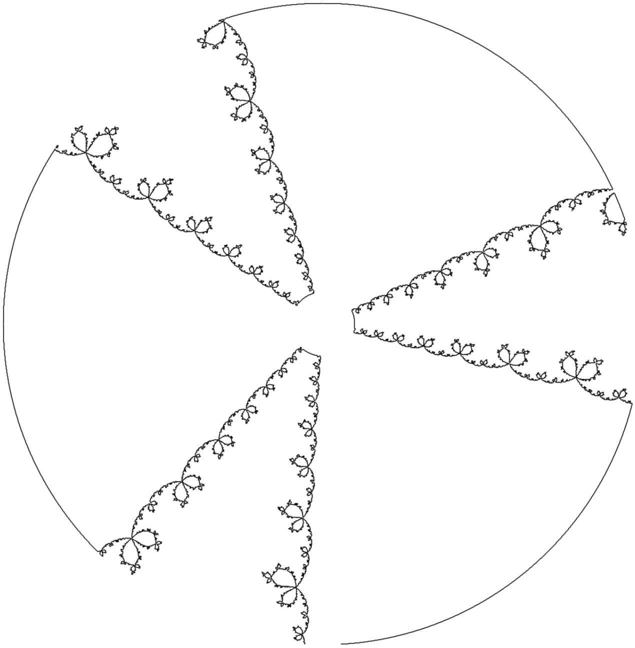
$$1 < \mathrm{ehypdim}(L) < \mathrm{HD}(J(L)) = 2.$$

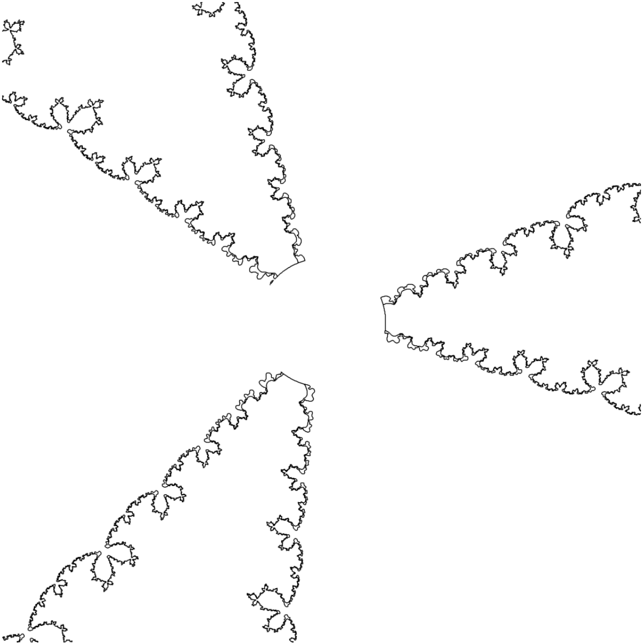
Corollary

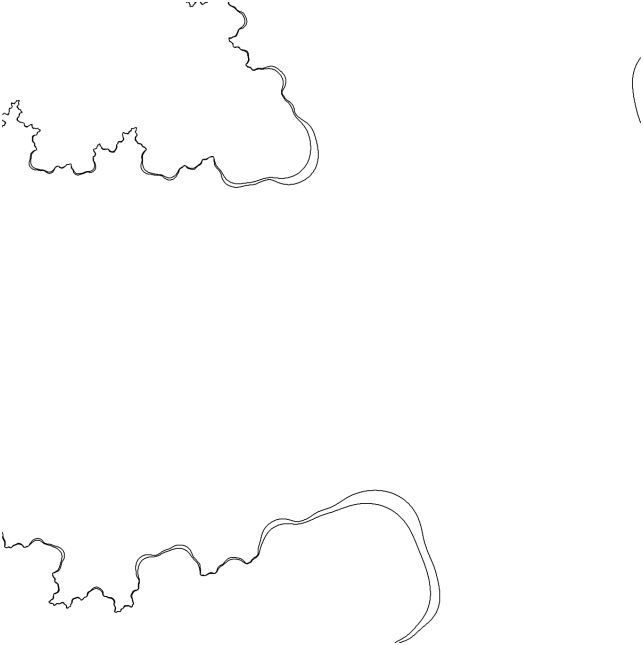
There exists a residual set R of the boundary of the Mandelbrot set, such that for all $c \in R$, and all repelling periodic cycle of $z^2 + c$, the corresponding Poincaré function L is in class $\mathcal{B}$, has finite positive order and satisfies

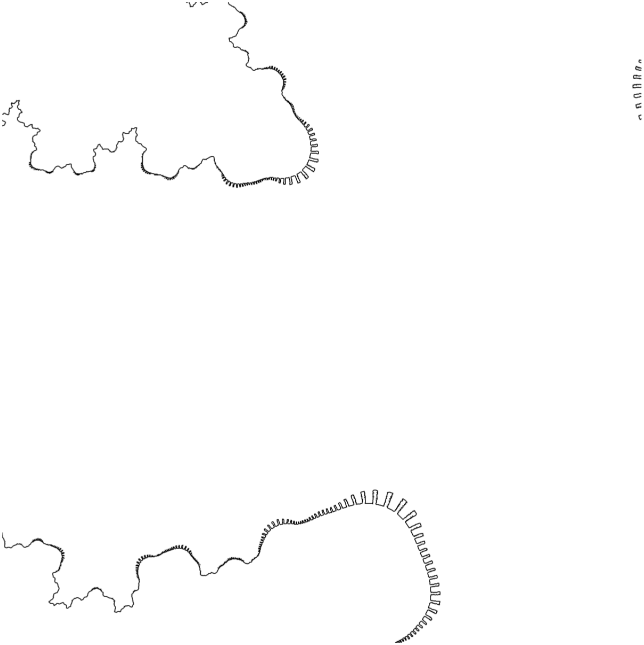
$$\mathrm{hypdim}(L) = 2.$$











Other linearisers

Theorem

For any entire function f,

$$\mathrm{ehypdim} L \geq \mathrm{hypdim}\, f.$$

Corollary

For $f(z) = 2\pi i e^z$, the Poincaré function L associated to $2\pi i$ have only finitely many singular values, and

$$\mathrm{hypdim}\, L = 2.$$

Moreover L can be chosen hyperbolic.

Equivalence classes and edim

Definition (Equivalences)

Two entire functions f and g are said to be **affinely equivalent** if there exists φ and ψ affine isomorphisms of $\mathbb{C}$ such that

$$\varphi f = g\psi.$$

Quasiconformal equivalence is the same with *affine* replaced by *quasiconformal*.

Proposition (Stallard, Rempe-Gillen)

The eventual dimension is constant along affine equivalence classes.

Equivalence classes and ehypdim

Proposition

The eventual hyperbolic dimension is constant along affine equivalence classes.

Theorem

There exists f and g class $\mathcal{B}$ functions of finite positive orders which are quasiconformally equivalent and satisfying

$$1 < \mathrm{ehypdim} f < \mathrm{ehypdim} g < 2.$$

THANK YOU !