# Escaping points in the boundaries of Baker domains

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Ergodic Theory and Holomorphic Dynamics
Erwin Schrödinger Institute, VIenna
September 29th, 2015

# Fatou components in the escaping set

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The complex plane decomposes into two totally invariant sets.

- The set of normality of the sequence $\{f^n\}_n$ is called the **The Fatou set (or stable set)**, denoted by $\mathcal{F}(f)$. It is open and its connected components are called **Fatou components**.
- **The Julia set (or chaotic set)**, $\mathcal{J}(f)$, is the complement of the Fatou set and closure of the set of repelling periodic points. Prepoles are dense in $\mathcal{J}(f)$.

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For special class of maps (e.g. finite type), the escaping set belongs to the Julia set.
But in general, there may exist Fatou components which belong to $\mathcal{I}(f)$.
Those are:

- (Some) **Wandering domains** $f^m(U) \cap f^n(U) = \emptyset$ for all $n, m$.
- (Some) **Baker domains** Periodic components (period $k$) for which $\{f^{nk}\}_n$ converge locally uniformly to $\infty$. All invariant $(k = 1)$ Baker domains are in $\mathcal{I}(f)$.

# Fatou components in the escaping set

## Natural questions

If $U$ is a Fatou component in $\mathcal{I}(f)$, natural questions are:

- Is $\partial U$ also in $\mathcal{I}(f)$?

Answer: Not in general.

Fatou's example

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- Can the opposite occur, i.e. $\partial U \cap \mathcal{I}(f) = \emptyset$?
- In general, how large (in terms of measure) is the set $\partial U \cap \mathcal{I}(f)$?

# Rippon and Stallard's results

These questions were first addressed by Rippon and Stallard.

**Theorem ([RS11], [RS14])**

*Let $f$ be an entire transcendental function and let $U$ be*

- *an escaping wandering domain, or*
- *a Baker domain satisfying $|f^{n+1}(z)| > K|f^n(z)|$ for some $z \in U$, $K > 1$ and all $n \geq 1$, or*
- *a Baker domain on which $f$ is univalent.*

*Let $\omega$ denote the harmonic measure on $\partial U$. Then, $\omega-$almost every point in $\partial U$ escapes.*


[RS11] P. J. Rippon and G. M. Stallard, *Boundaries of escaping Fatou components*, Proc. Amer. Math. Soc. 139 (2011), no. 8, 28072820.

[RS14] P.J. Rippon and G. Stallard, *Boundaries of univalent Baker domains*. To appear in J. Anal. Math., 2014

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To answer these questions we need to briefly introduce

- Inner functions and singularities;
- Classification of Baker domains

# Inner functions

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- Clearly

$$\deg(g) = \deg(f|_U),$$

  and $g$ has no fixed points in $\mathbb{D}$.

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- By the Denjoy-Wolff Theorem, there exists $p \in \partial\mathbb{D}$ such that

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  locally uniformly on $\mathbb{D}$. The point $p$ is the **Denjoy-Wolff point** of $g$.
- We say that $f|_U$ is **regular** if $p$ is **NOT a singularity of $g$**.

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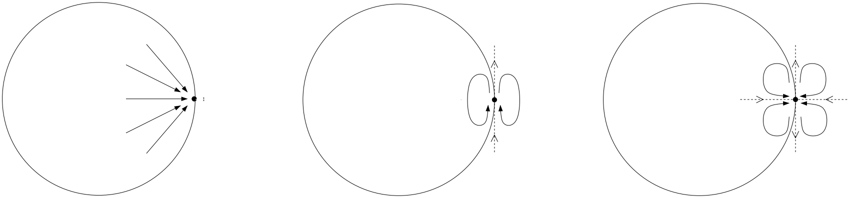
- BUT, the asymptotic dynamics of $g$ on $\partial\mathbb{D}$ do not need to correspond with those of $f$ on $\partial U$ because of the discontinuities of $\varphi$ on $\partial\mathbb{D}$.

# Classification of Baker domains

Baker-Pommerenke-Cowen // König

There exists an absorbing domain $W \subset U$, and the dynamics in $W$ are conformally conjugate to either

$$\begin{cases} a\,\omega, \quad a > 1 & \text{in } \mathbb{H} \quad \textbf{hyperbolic type} \\ \omega \pm i & \text{in } \mathbb{H} \quad \textbf{simply parabolic type} \\ \omega + 1, & \text{in } \mathbb{C} \quad \textbf{doubly parabolic type}. \end{cases}$$



hyperbolic  simply parabolic  doubly parabolic

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Classifying particular Baker domains is not an easy task. Some geometric characterizations exist in [FH06] in terms of $U/f$ and in [BFJK14] in terms of the hyperbolic distance between iterates. The following is relevant for our setting.

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Let $f$ and $U$ be as above and let $\rho_U$ denote the hyperbolic distance in $U$.

**Theorem ([BFJK14])**

*$f|_U$ is doubly parabolic* $\iff$ $\rho_U(f^{n+1}(z), f^n(z)) \underset{n\to\infty}{\longrightarrow} 0$ *for some $z \in U$*

[BFJK14] K.Barański, N.F, X.Jarque and B.Karpińska, *Absorbing sets and Baker domains for holomorphic maps*, J. of the LMS **92** (2015), 144-162.

# Main Results

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We first see that in many cases, Rippon and Stallard's theorem remains true.

**Theorem A**

*Suppose*

- $f|_U$ *is hyperbolic or simply parabolic (i.e.* $\rho_U(f^{n+1}(z), f^n(z)) \nrightarrow 0$*), and*
- $f|_U$ *is regular (e.g. if* $\deg(f) < \infty$*).*

*Then,* $\omega-$*almost every point in* $\partial U$ *escapes.*

We remark that if $f|_U$ is univalent, then it is always hyperbolic or simply parabolic.

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### Theorem B

*Suppose*

- $f|_U$ *is doubly parabolic (i.e.* $\rho_U(f^{n+1}(z), f^n(z)) \longrightarrow 0$*), and*
- $\deg(f) < \infty$.

*Then,* $\omega-$*almost every point in* $\partial U$ *is topologically recurrent and, in particular, it does NOT escape.*

A point $z \in \mathbb{C}$ is *topologically recurrent* under $f$ if its orbit visits any neighborhood of $z$ infinitely often.

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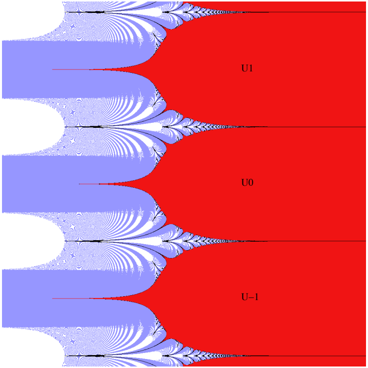
In fact, more is true: $f$ is recurrent with respect to $\omega$.

# An Example for Theorem B

Consider $f(z) = z + e^{-z}$ (Newton's map of $e^{-e^z}$).




- $f$ has infinitely many Baker domains, of degree 2, doubly parabolic.
- Hence the set of escaping points in $\partial U_i$ has harmonic measure 0.
- We conjecture that
  - all escaping points are nonaccessible from $U_i$, while
  - accessible periodic points are dense in $\partial U_i$.

# Remarks about Theorem B

- The hypothesis of finite degree CANNOT be removed.
  - Aaronson'78 and Doering-Mañé'91 give an example of a simply connected Baker domain, of infinite degree, of doubly parabolic type for which $\omega-$almost every point in the boundary escapes.

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- On the other hand, there exist Baker domains of ininite degree for which Theorem B holds.
    - Roughly speaking, that happens when

$$\rho_U(f^n(z), f^{n+1}(z)) \longrightarrow 0 \quad \textbf{fast enough},$$

even though it is always the case that

$$\sum_{n=0}^{\infty} \rho_U(f^n(z), f^{n+1}(z)) = \infty.$$

# A refinment

The precise condition is as follows, with **no assumption on the degree of $f|_U$.**

### Theorem C

*Let $f$ be a meromorphic transcendental map and $U$ a simply connected invariant Baker domain such that*

$$\rho_U(f^n(z), f^{n+1}(z)) \leq \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^r}\right),$$

*as $n \to \infty$ for some $z \in U$ and $r > 1$. Then, $\omega-$almost all boundary points are topologically recurrent. In particular, non-escaping points have full harmonic measure.*

## A remark about simply connected parabolic basins

- Suppose $U$ is an invariant simply connected parabolic basin, i.e. such that $f^n \to \zeta \in \partial U \cap \mathbb{C}$ locally uniformly on $U$, and $f'(\zeta) = 1$.

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$$\mathcal{I}_P(f) = \{z \in \partial U \mid f^n(z) \underset{n\to\infty}{\longrightarrow} \zeta\}.$$

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- Using extended Fatou coordinates, one can see that $U$ is of doubly parabolic type in the sense of Baker-Pommerenke-Cowen.
- Then, Theorems B and C remain valid in this setting. In fact, for rational maps, Theorem B was proven in Doering-Mañé'91, and Aaronson-Denker-Urbanski 93.

[DM91] Claus I. Doering and Ricardo Mañe, *The dynamics of inner functions.*, Rio de Janeiro: Sociedade Brasileira de Matemática, 1991.

[UA93] J.Aaronson, M.Denker and M.Urbanski, *Ergodic theory for Markov fibred systems and parabolic rational maps*,

Trans. Amer. Math. Soc. **337** (1993), 495-548.

# A question remains...

Recall the statement of Theorem C.

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**Question:** Are there actually Baker domains satisfying the asumptions??

# A family of examples

**Answer:** Yes, and in fact there is a whole family of examples.

## Proposition D

*Let $f$ be a meromorphic map of the form*

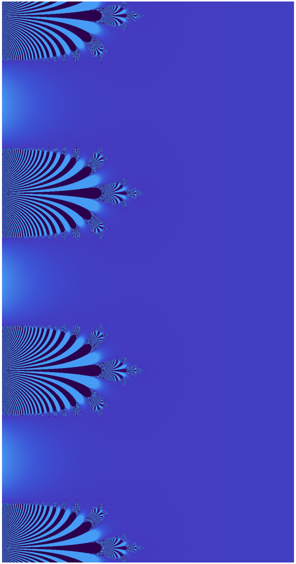
$$f(z) = z + a + h(z)$$

*where $a \in \mathbb{C} \setminus \{0\}$ and*

$$|h(z)| < \frac{c_0}{(\mathrm{Re}(z/a))^r} \quad \textit{for} \quad \mathrm{Re}\left(\frac{z}{a}\right) > c_1, \quad r > 1, c_0, c_1 > 0.$$

*Then $f$ has an invariant Baker domain $U$ containing a half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z/a) > c\}$ for some $c \in \mathbb{R}$. Moreover, if $U$ is simply connected (e.g. if $f$ is entire), then $f$ on $U$ satisfies the assumptions of Theorem $\mathrm{C}$ and, consequently, $\omega-$almost every point is non-escaping.*

# Example 1: Fatou's example

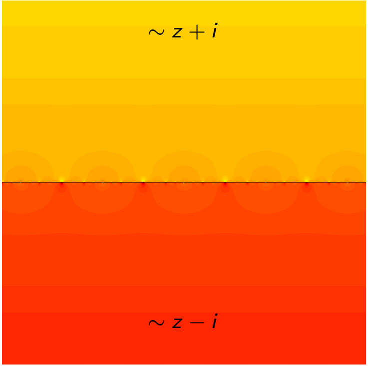


- $f(z) = z + 1 + e^{-z}$ has a Baker $U$ domain which contains $\{\mathrm{Re}(z) > 1\}$.

- The degree of $f|_U$ is infinite.

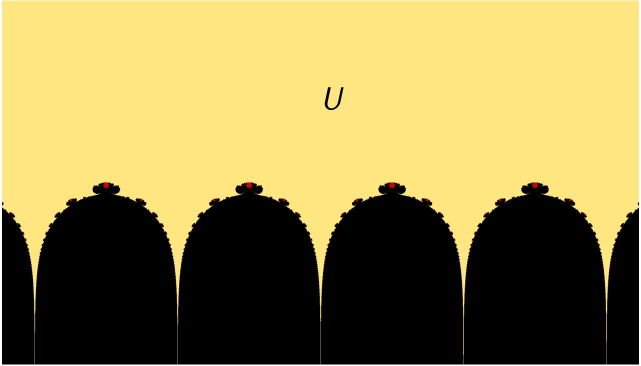
- $f$ satisfies the hypothesis of Proposition D (for $a = 1$), hence the Baker domain is doubly parabolic and $\omega-$almost every point in $\partial U$ is topologically recurrent.

# Example 2 ([DM91] and [BFJK15])




- $f(z) = z + \tan z$ has two Baker domains $U+ = \{\mathrm{Im}(z) > 0\}$ and $U_- = \{\mathrm{Im}(z) < 0\}$.
- The degree of $f|_{U_\pm}$ is infinite, but $f$ is not regular.
- $f$ satisfies the hypothesis of Proposition D (for $a = \pm i$), hence the Baker domain is doubly parabolic and $\omega-$almost every point in $\partial U$ is topologically recurrent.

# Example 3: $f(z) = z + i + \tan z$




- (Yellow) Baker domain $U$ of infinite degree 2. Satisfies the hypothesis of Prop. D for $a = 2i$.
- (Black) Infinitely many Baker domains of degree 2, doubly parabolic. Satisfy Theorem B.

# Proof of Theorem A (sketch)

Recall that $\varphi : \mathbb{D} \to U$ is a Riemann map, $g : \mathbb{D} \to \mathbb{D}$ is the inner function and $p$ the Denjoy-Wolff point.

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   - This is obvious in the univalent case but here one must work locally and extend later.
2. Prove that almost all of these points map under $\varphi$ to escaping points.
   - Here we must use a Pflüger type estimate on the behaviour of Riemann maps. This (great) idea is from [RS14].

# Tool 1: A dichotomy

**Theorem[DM91]**

Let $g : \mathbb{D} \to \mathbb{D}$ be an inner function. Then the following hold.

(a) If $\sum_{n=1}^{\infty}(1 - |g^n(z)|) < \infty$ for some point $z \in \mathbb{D}$, then $g^n$ converges to a point $p \in \partial\mathbb{D}$ almost everywhere on $\partial\mathbb{D}$.

(b) Otherwise $g$ is recurrent on $\partial\mathbb{D}$ with respect to the Lebesgue measure.

# Tool 2

This is a quantitative estimate of the principle that *"sets that are difficult to reach have very small harmonic measure"*.

## Theorem [Pomm92]

Let $\Phi : \mathbb{D} \to \mathbb{C}$ be a conformal map, let $V \subset \Phi(\mathbb{D})$ be a non-empty open set and let $E$ be a Borel subset of $\partial\mathbb{D}$. Suppose that there exist $\alpha \in (0, 1]$ and $\beta > 0$ such that:

(a) $\text{dist}(\Phi(0), V) \geq \alpha|\Phi'(0)|$,

(b) $\ell(\Phi(\gamma) \cap V) \geq \beta$ for every curve $\gamma \subset \mathbb{D}$ connecting 0 to $E$.

Then,

$$\lambda(E) < \frac{15}{\sqrt{\alpha}}\, e^{-\frac{\pi\beta^2}{\text{area} V}}.$$

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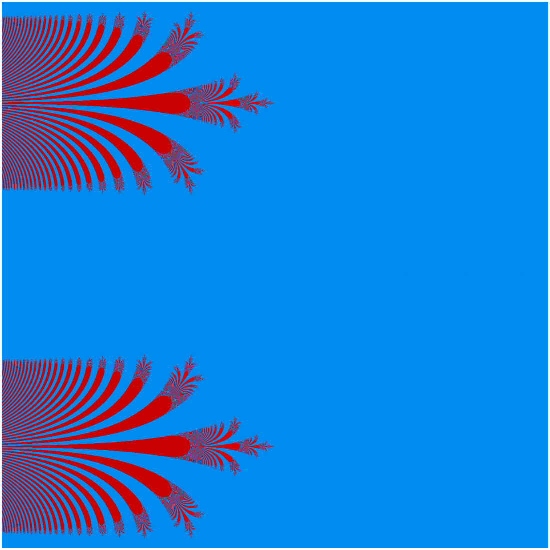
4. $g$ recurrent implies that $\omega-$almost every point in $\partial U$ is topologically recurrent.

q.e.d.

# Thank you for your attention!

# Fatou's Example

Example: $z \mapsto z + 1 + \exp(-z)$.



$z = 2k\pi i, \; k \in \mathbb{Z}$

are repelling fixed points in $\partial U$ and hence nonescaping.

Back to questions