Connectedness properties of $I^+(f)$

Note Title 11/06/2015

f is a transcendental entire function

Joint work with John Osborne & Gwyneth Stallard

$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$

Eremenko 1989: $I(f) \cap J(f) \neq \emptyset$, $J(f) = \partial I(f)$

E's conjecture: all components of $I(f)$ are unbounded

Fast escaping set: $A(f) = \bigcup_{\ell \geq 0} f^{-\ell}(A_R(f))$

$A_R(f) = \{z : |f^n(z)| \geq M^n(R), \text{ for } n \geq 0\}$

where $M(r) = \max_{|z|=r} |f(z)|$ and $M(r) > r$, for $r \geq R$.

Thm 1 (R+S 2005) All components of $A(f)$ are unbounded.

Baker's conjecture: if f has order $\rho < \frac{1}{2}$, then all components of $F(f)$ are bounded.

True for periodic case (Stallard, Anderson + Hinkkanen, Zheng).

Thm 2 (R+S, Nicks+R+S) Baker's conjecture holds if
(a) f has very small growth
(b) $\rho < \frac{1}{2}$ and $M(r)$ is 'regular'
(c) f is real with real zeros and $\rho < \frac{1}{2}$.

Cases (a) & (b) : proofs show $A_R(f)$ is a 'spider's web.'

Case (c) : $A_R(f)$ may not be a SW, but $Q_{\varepsilon,R}(f)$ contains one.

Quite fast escaping set: $Q(f) = \bigcup_{0<\varepsilon<1} Q_\varepsilon(f)$

$$Q_\varepsilon(f) = \bigcup_{\ell \geq 0} f^{-\ell}(Q_{\varepsilon,R}(f))$$

$$Q_{\varepsilon,R}(f) = \{z : |f^n(z)| \geq \mu_\varepsilon^n(R),\ n \geq 0\},$$

where $\mu_\varepsilon(r) = M(r)^\varepsilon$ and $\mu_\varepsilon(r) > r$, for $r \geq R$.

Hypotheses of Thm 2(c) imply: $Q(f)$ is a SW.

This suggests

$\boxed{\text{Conjecture: } \rho < \tfrac{1}{2} \Rightarrow Q(f) \text{ is a SW } (\Rightarrow I(f) \text{ is a SW})}$

Method of proof would solve Baker's conjecture...

Thm 3 (Osborne + R+S)
$\varrho < \frac{1}{2} \Rightarrow Q^+(f)$ is a weak SW.

Weak spider's web: E is connected and E^c contains no unbounded closed connected set.

Meaning of +:
condition on z holds on a subsequence (depending on z).

e.g.

$$I^+(f) = \{z : \exists (n_j) \text{ s.t. } f^{n_j}(z) \to \infty \text{ as } j \to \infty\}$$

$$Q^+(f) = \{z : \exists \varepsilon, \ell, (n_j) \text{ s.t. } |f^{n_j+\ell}(z)| \geq \mu_\varepsilon^{n_j}(R),\ j \geq 0\}.$$

Minimum modulus : $m(r) = m(r,f) = \min_{|z|=r} |f(z)|$

For $\rho < \frac{1}{2}$, the $\cos\pi\rho$ theorem gives :

$$m(r) \sim M(r)^{\cos\pi\rho}, \quad \text{most of the time},$$

so

$\exists\, r > 0$ s.t. $m^n(r) \to \infty$ as $n \to \infty$,

moreover, $m^n(r) \geq \mu_\varepsilon^n(R)$, $n \geq 0$. $\varepsilon \sim \cos\pi\rho$

$\boxed{\begin{array}{l}\text{Idea : consider} \\ \bullet \text{ functions } f \text{ s.t. } m^n(r) \to \infty \text{ for some } r > 0 \\ \bullet \text{ set of } z \text{ s.t. } |f^n(z)| \geq m^n(r),\ n \geq 0.\end{array}}$

$I(f,(a_n)) = \{ z : |f^n(z)| \geq a_n, \text{ for } n \in \mathbb{N} \}.$

$$I(f,(a_n)) = \{z : |f^n(z)| \geq a_n, \text{ for } n \in \mathbb{N}\}.$$

Thm 4 (ORS) Suppose $m^n(r) \to \infty$.

The following are all weak SWs:

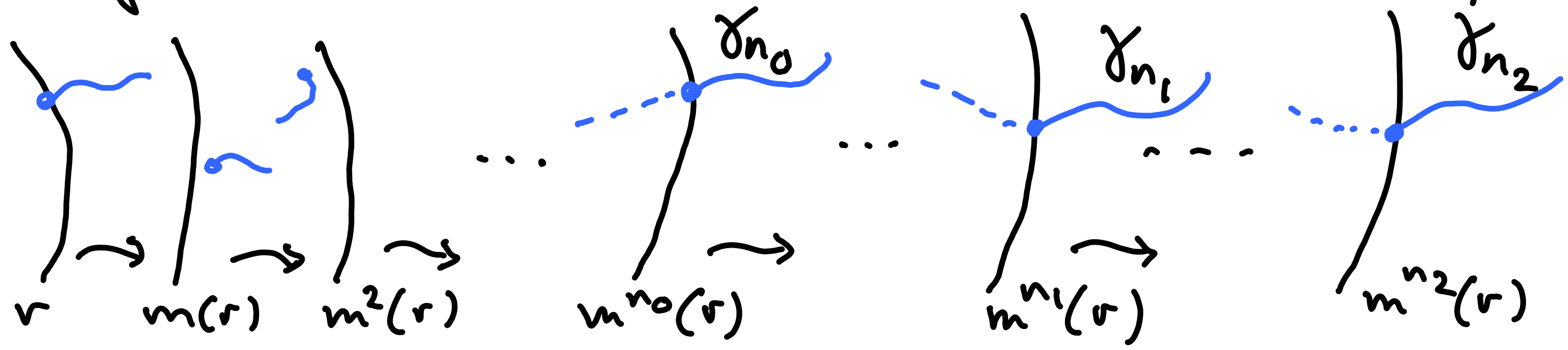
(a) $I^+(f)$

(b) $I^+(f,(m^n(r))) = \{z : \exists (n_j) \text{ s.t. } |f^{n_j}(z)| \geq m^{n_j}(r),\ j \geq 0\}$

(c) $\bigcup_{\ell \geq 0} f^{-\ell}(I^+(f,(m^n(r))))$.

$\Rightarrow$ Thm 3

'**Proof**' Use contradiction with continua in complement.



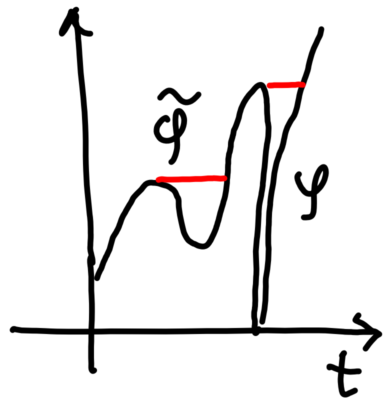
How to check if : $\exists\, r$ s.t. $m^n(r) \to \infty$ as $n \to \infty$.

Clearly not if $f \in \mathcal{B}$ – more generally if f is bounded on a curve to ∞.

Also, $m(r)$ unbounded, or even $m(r)/r$ unbounded, is not sufficient.

Thm 5 (ORS) Let $\varphi : [0,\infty) \to [0,\infty)$ be cts and

$$\tilde{\varphi}(t) = \max\{\varphi(s) : 0 \le s \le t\}.$$



TFAE

(a) $\exists\, t$ s.t. $\varphi^n(t) \to \infty$ as $n \to \infty$

(b) $\exists\, T$ s.t. $\tilde{\varphi}(t) > t$, for $t \ge T$.

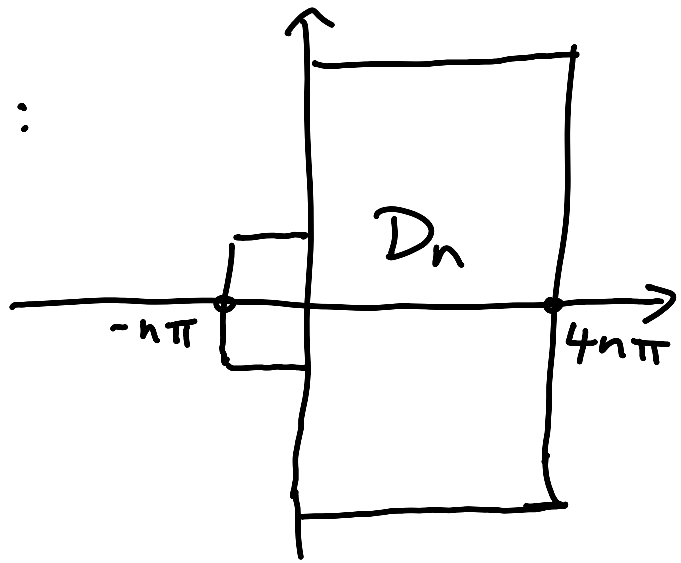
Example: $f(z) = 2z + e^{-z}$, for each $k \in \mathbb{N}$, $r = (2k+1)\pi$ satisfies $m(r) \geq \frac{4}{3} r$.

Generalisation: $\exists$ domains D_n, $n \geq 0$, s.t.

- $f(\partial D_n)$ surrounds D_{n+1}
- every disc Δ in $\mathbb{C}$ is contained in D_n, for $n \geq N(\Delta)$.

Covers e.g. $f(z) = -\frac{1}{2}z - 10ze^{-z}$

$m(r) \sim \frac{1}{2} r$, but can take:



What other properties of $I(f)$ generalise?

Thm 6 (R+S 2013)

(a) If U is a bounded s-conn domain:

$$U \cap I(f) \neq \emptyset \implies \partial U \cap I(f) \neq \emptyset.$$

(b) $\underline{I}(f) \cup \{\infty\}$ is connected.

Proof depends on:

· if U is a bounded wandering domain in $I(f)$, then

$$\partial U \cap I(f) \neq \emptyset.$$

In fact

$$\partial U \cap \underline{I}(f) \text{ has full harmonic measure.}$$

Thm 6 holds with $I(f)$ replaced by:

$$I^+(f),\ Q(f),\ Q^+(f), \ldots$$

Questions

(a) If $I(f)$ is disconnected does it have uncountably many components?

Yes for $J(f)$ and yes for $I^+(f)$!

(b) Is it true that if U is a bounded wandering domain in $I(f)$, then
$$\partial U \subset I(f)?$$

Same question for $I^+(f)$.